Fidelity susceptibility in Gaussian Random Ensembles

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Setting

- Parametric family of Hamiltonians

$$H = H_0 + \lambda H_1$$

- Pure state fidelity

$$\mathcal{F} = |\langle \psi(0) | \psi(\lambda) \rangle|$$

- We are interested in fidelity of eigenstates: $H(\lambda)|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle$

- For small λ

$$\mathcal{F} = 1 - \frac{1}{2}\chi\lambda^2 + O(\lambda^3)$$

  (first order terms vanish due to the normalization of eigenfunctions)

- **Definition:** χ - ***fidelity susceptibility*** (of H_0)

Properties

- Alternatively
$$\chi = \frac{\partial^2 \mathcal{F}}{\partial \lambda^2}\Big|_{\lambda=0}$$
- Hence
$$\chi = \left(\langle \partial_\lambda \psi(\lambda) | \partial_\lambda \psi(\lambda)\rangle - \langle \psi(\lambda) | \partial_\lambda \psi(\lambda)\rangle - \langle \partial_\lambda \psi(\lambda) | \psi(\lambda)\rangle\right)\big|_{\lambda=0}$$

  what exhibits a nice geometric picture - the real part of the natural Riemannian structure on a manifold of quantum states (Provost and Vallee, 1980)

- Quantum Fisher Information G
$$G = 4\chi$$
- The n-th eigenastate fidelity susceptibility for H_0 can be alternatively expressed as
$$\chi_n = \sum_{m \neq n} \frac{|H_{1,nm}|^2}{(E_n - E_m)^2}$$
$$H_{1,nm} = \langle \psi_n(0) | H_1 | \psi_n(0)\rangle, \quad E_n = E_n(0)$$

Applications

- Quantum phase transitions
  - Bose-Hubbard model
    (You, Li, Gu, Phys. Rev. E **76**, 022101, 2007)
  - XY model
    (Zanardi, Paunković, Phys. Rev. E **74**, 031123, 2006)
  - Dicke model
    (*ibid.*)
- Quantum many-body localization
  (Hu *et al.*, Phys. Rev. E **94**, 052119, 2016;
  Maksymov, Sierant, Zakrzewski, *in preparation*.)
- Whenever you know an ingenuous application of the Quantum Fisher Information

Random Matrices

- Usually quite complicated Hamiltonians. A minimalistic assumption
- H_a, $a = 1, 2$ are random matrices from the classical Random $(N \times N)$ Matrix Gaussian ensembles with densities

$$P(H_a) \sim \exp\left(-\frac{\beta}{4J^2}\mathrm{Tr}H_a^2\right)$$

  - variance

$$\langle H_{nn}^2 \rangle = 2\langle H_{mn}^2 \rangle = 2J^2$$

  - GOE ($\beta = 1$), GUE ($\beta = 2$), GSE ($\beta = 4$)
  - ultimately $N \to \infty$, $J = O(1/N)$

Detour. Quantum Chaos. Level Curvature Distribution

- Similar quantity was thoroughly investigated in the context of disordered system and quantum chaos
- Level Curvature Distribution

$$K_n := \frac{\partial E_n(\lambda)}{\partial \lambda} = -E_n + \sum_{m \neq n} \frac{|H_{1,nm}|^2}{E_m - E_n}$$

  - Characterization of spectral fluctuations in quantum chaotic systems
  - Conductance in disordered systems $\langle g \rangle \sim \langle |K| \rangle$, ($\lambda$ - magnetic flux through the probe)
- The distribution of the curvature in quantum chaotic or disordered systems in Random Matrix Theory, as conjectured by Zakrzewski and Delande

$$W(K) \sim (1 + K^2)^{1+\beta/2}$$

- Later proved by von Oppen and Fyodorov & Sommers

Back to susceptibility

Our task: the distribution of susceptibility at the energy E

$$P(\chi, E) = \frac{1}{N\rho(E)} \left\langle \sum_{n=1}^{N} \delta(\chi - \chi_n)\delta(E - E_n) \right\rangle$$

where the averaging $\langle\ \rangle$ is over

$$P(H_0, H_1) \sim \exp\left[-\frac{\beta}{4J^2}\left(\mathrm{Tr}H_0^2 + \mathrm{Tr}H_1^2\right)\right]$$

and $\rho(E)$ - the density of states

Calculations. Few tricks

- Fourier representation of $\delta(\chi - \chi_n)$
- averaging over H_1 reduces to a Gaussian integral
- averaging over H_0 reduces to the one over the distribution of eigenvalues only, i.e. with the distribution

$$P(E_1, E_2, \ldots, E_N) \sim \prod_{k<l} |E_k - E_l|^\beta \, \mathrm{e}^{-\frac{\beta}{4J^2} \sum_k E_k^2}$$

- integrating with $\delta(E - E_n)$ and using the orthogonal/unitary invariance of the RMT distributions allows reducing the dimension N by 1
- at the center of the spectrum $E = 0$ (can be relaxed) we arrive at

$$P(\chi) \sim \int_{-\infty}^{\infty} d\omega e^{-i\omega\chi} \left\langle \left[\frac{\det \bar{H}^2}{\det \left(\bar{H}^2 - \frac{2i\omega J^2}{\beta} \right)^{\frac{1}{2}}} \right]^\beta \right\rangle_{N-1}$$

where the averaging is now over $\bar{H}$ from $(N-1) \times (N-1)$ ensemble

Calculations. Some further tricks

- Gaussian integral representation

$$\det\left(\bar{H}^2 - \frac{2i\omega J^2}{\beta}\right)^{-\frac{\beta}{2}} \sim \int d\mathbf{z}\, \exp\left[-\mathbf{z}^\dagger\left(\bar{H}^2 - \frac{2i\omega J^2}{\beta}\right)\mathbf{z}\right]$$

  where $\mathbf{z}$, a $N-1$-component real/complex vector, due to orthogonal/unitary invariance may be chosen as $\mathbf{z} = r[1, 0, 0..]^T$

- We arrive at

$$P(\chi) \sim \int_0^\infty dr r^s \delta\left(\chi - 2J^2 r^2/\beta\right) \left\langle \det \bar{H}^{2\beta} e^{-r^2 X}\right\rangle_{N-1}$$

- Block matrix representation

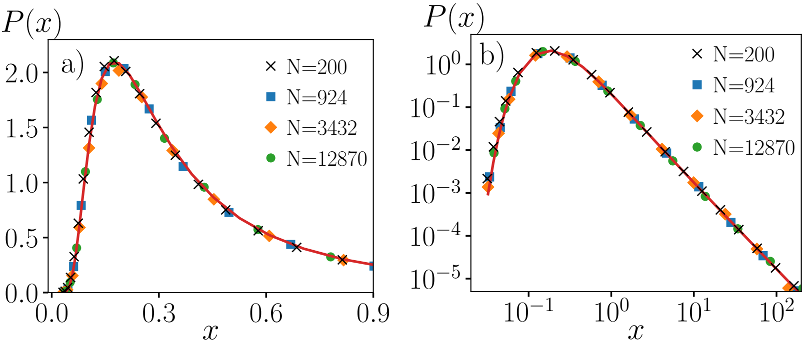
$$\bar{H} = \begin{bmatrix} H_{11} & H_{1j} \\ H_{1k} & V \end{bmatrix}$$

  and integration over H_{1m} leaves the averaging over the $(N-2)\times(N-2)$ GOE/GUE matrix V

Asymptotic results GOE

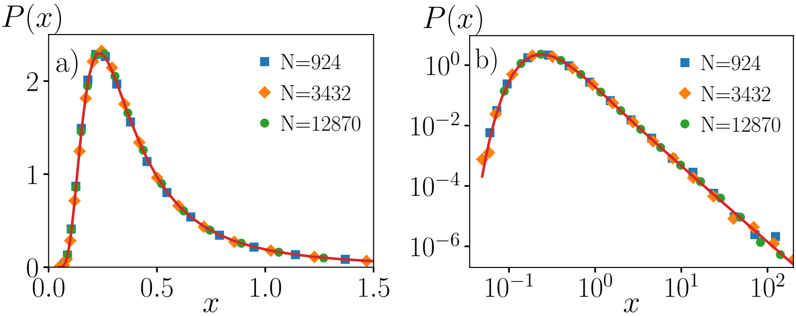
The asymptotic ($N \to \infty$) results for the scaled fidelity susceptibility $x = \chi/N$

$$P^O(x) = \frac{1}{6}\frac{1}{x^2}\left(1 + \frac{1}{x}\right)\exp\left(-\frac{1}{2x}\right)$$



Asymptotic results GUE

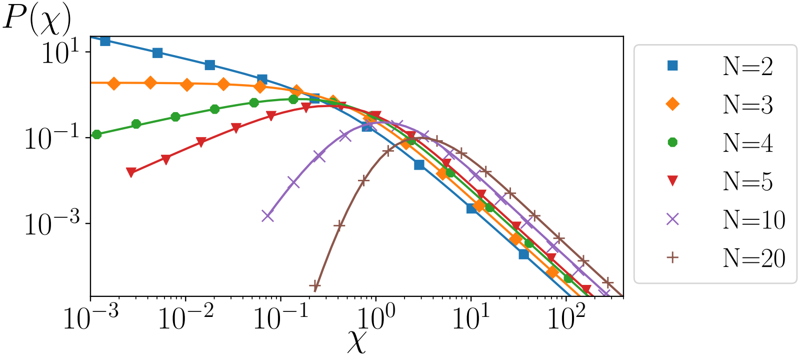
$$P^U(x) = \frac{1}{3\sqrt{\pi}} \frac{1}{x^{5/2}} \left(\frac{3}{4} + \frac{1}{x} + \frac{1}{x^2} \right) \exp\left(-\frac{1}{x} \right)$$



Arbitrary N. GOE

$$P_N^O(\chi) = \frac{C_N^O}{\sqrt{\chi}} \left(\frac{\chi}{1+\chi}\right)^{\frac{N-2}{2}} \left(\frac{1}{1+2\chi}\right)^{\frac{1}{2}} \left[\frac{1}{1+2\chi} + \frac{1}{2}\left(\frac{1}{1+\chi}\right)^2 \mathcal{I}_{N-2}^{O,2}\right]$$

$$\mathcal{I}_N^{O,2} = \begin{cases} N\frac{N+2}{N+3/2}, & N \quad \text{even}, \\ N+1/2, & N \quad \text{odd}. \end{cases}$$



Arbitrary N. GUE

$$
\begin{aligned}
P_N^U(\chi) &= C_N^U \left(\frac{\chi}{1+\chi}\right)^{N-2} \left(\frac{1}{1+2\chi}\right)^{\frac{1}{2}} \times \\
&\times \left[\frac{3}{4}\left(\frac{1}{1+2\chi}\right)^2 + \frac{3}{2}\frac{1}{1+2\chi}\left(\frac{1}{1+\chi}\right)^2 \mathcal{I}_{N-2}^{U,2} + \frac{1}{4}\left(\frac{1}{1+\chi}\right)^4 \mathcal{I}_{N-2}^{U,4}\right]
\end{aligned}
$$

$$
\mathcal{I}_N^{U,2} = \begin{cases} \frac{1}{3}N, & N \text{ even}, \\ \frac{1}{3}(N+1), & N \text{ odd}, \end{cases} \qquad \mathcal{I}_N^{U,4} = \begin{cases} N^2 + 2N, & N \text{ even}, \\ N^2 + 4N + 3, & N \text{ odd}. \end{cases}
$$