On the geometry of one-mode Gaussian channels

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Contents

Gaussian states

Consider the n-particle continuous variable system.

- Infinite-dimensional Hilbert space $\mathcal{H} = \bigotimes_{k=1}^{n} L^2(\mathbb{R})$.
- Vector states $R = (q_1, p_1, \ldots, q_n, p_n)^T$ satisfying the commutation relations

$$[R_i, R_j^\dagger] = 2i\Omega_{ij}, \qquad \Omega = \bigoplus_{k=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1}$$

  with the symplectic form Ω.
- Density operators are given by

$$\rho = \int_{\mathbb{R}^{2n}} \frac{\mathrm{d}^{2n}\xi}{\pi^n} \chi(\xi) D(-\xi) \tag{2}$$

  with the displacement (Weyl) operators

$$D(\xi) = e^{iR^T\Omega\xi}. \tag{3}$$

Gaussian states

Definition 1

A density operator

$$\rho = \int_{\mathbb{R}^{2n}} \frac{d^{2n}\xi}{\pi^n} \chi(\xi) D(-\xi) \tag{4}$$

is a Gaussian state if its characteristic function $\chi(\xi)$ is a Gaussian function.

A Gaussian characteristic function is represented by

$$\chi(\xi) = \exp\left[-\frac{1}{2}\xi^T \Omega \Sigma \Omega^T \xi + i\ell^T \Omega \xi\right], \tag{5}$$

where

- $\ell_k = \mathrm{Tr}[\rho R_k]$ is the displacement vector;
- $\Sigma_{ij} = \frac{1}{2}\mathrm{Tr}[\rho(R_i R_j + R_j R_i)] - \ell_i \ell_j$ is the covariance matrix.

Σ is the covariance matrix of a Gaussian state if and only if

$$\Sigma + i\Omega \geq 0. \tag{6}$$

Gaussian channels

Definition 2

A Gaussian channel Λ is a quantum channel that transforms Gaussian states into Gaussian states.

- Representation in the Heisenberg picture:

$$\Lambda^*[D(\xi)] = D(M\xi)\exp\left[-\frac{1}{2}\xi^T N\xi + ic^T\xi\right]. \tag{7}$$

- Each channel is completely characterized by a triple (M, N, c), and it acts on the Gaussian state $\rho(\Sigma, \ell)$ as follows,

$$\Sigma \mapsto M^T\Sigma M + N, \qquad \ell \mapsto M^T\ell + c. \tag{8}$$

- The complete positivity condition:

$$N - iM^T\Omega M + i\Omega \geq 0. \tag{9}$$

Choi-Jamiołkowski isomorphism

Theorem 3

There exists a one-to-one correspondence between the bipartite Gaussian states ρ_{AB} with a common marginal $\sigma = \mathrm{Tr}_A \, \rho_{AB}$ and the Gaussian channels $\Lambda : \mathcal{H}_B \to \mathcal{H}_A$, such that

$$\rho_{AB} = (\Lambda \otimes \mathbb{I}_B)(\rho_\Omega), \qquad (10)$$

where the Gaussian state ρ_Ω is characterized by

$$\Sigma_\Omega = \begin{pmatrix} \Sigma_\sigma & S_\sigma^T Z_\sigma S_\sigma \\ S_\sigma^T Z_\sigma S_\sigma & \Sigma_\sigma \end{pmatrix}, \qquad \ell_\Omega = \ell_\sigma \oplus \ell_\sigma. \qquad (11)$$

J. Kiukas, C. Budroni, R. Uola, and J.-P. Pellonpää, Phys. Rev. A **96***, 042331 (2017).*

- S_σ is the symplectic matrix diagonalizing Σ_σ;
- $Z_\sigma = \bigoplus_{k=1}^{n} \sigma_3 \sqrt{\nu_{\sigma,k}^2 - 1}$;
- $\nu_{\sigma,k}$ are the symplectic eigenvalues of Σ_σ.

Line and volume element

The Hilbert-Schmidt distance is defined by $\mathrm{d}\,s^2 = \mathrm{Tr}(\mathrm{d}\,\rho^2)$.

- For the Gaussian states $\rho(\Sigma, \ell = 0)$, one has

$$\mathrm{d}\,s^2 = \frac{1}{16\sqrt{\det \Sigma}} \left\{ 2\,\mathrm{Tr}(\Sigma^{-1}\,\mathrm{d}\,\Sigma)^2 + [\mathrm{Tr}(\Sigma^{-1}\,\mathrm{d}\,\Sigma)]^2 \right\}. \quad (12)$$

- The volume element corresponding to $\mathrm{d}\,s^2 = \mathrm{d}\,\boldsymbol{\Sigma}^T G\,\mathrm{d}\,\boldsymbol{\Sigma}$ reads

$$\mathrm{d}\,V = \sqrt{\det G} \prod_{k=1}^{4n^2} \mathrm{d}\,\boldsymbol{\Sigma}_k, \quad (13)$$

  where $\mathrm{d}\,\boldsymbol{\Sigma} = \mathrm{vec}\,\mathrm{d}\,\Sigma$, and G is the metric.

Link and W. T. Strunz, J. Phys. A: Math. Theor. **48**, *275301 (2015).*

Line and volume element

Consider the one-mode Gaussian channels ($n = 1$), which correspond to the two-mode CJ Gaussian states with

$$\begin{cases} \Sigma &= \begin{pmatrix} \Sigma_A & \Gamma^T \\ \Gamma & \Sigma_\sigma \end{pmatrix}, \\ \ell &= \ell_A \oplus \ell_\sigma, \end{cases} \qquad \begin{cases} \Sigma_A &= N + M^T \Sigma_\sigma M, \\ \Gamma &= S_\sigma^T Z_\sigma S_\sigma M, \\ \ell_A &= c + M^T \ell_\sigma. \end{cases} \tag{14}$$

- The purity-seralian coordinates:

$$\mu = \frac{1}{\sqrt{\det \Sigma}}, \quad \mu_{A/\sigma} = \frac{1}{\sqrt{\det \Sigma_{A/\sigma}}}, \quad \Delta = \det \Sigma_A + \det \Sigma_\sigma + 2 \det \Gamma. \tag{15}$$

- The volume element:

$$\mathrm{d}\, V = \frac{\mu^{11/2}}{64\sqrt{2}\mu_A^3 \mu_\sigma^2} \,\mathrm{d}\,\mu_A \,\mathrm{d}\,\mu \,\mathrm{d}\,\Delta \,\mathrm{d}\,\theta \,\mathrm{d}\, m(S_A), \tag{16}$$

where $\mathrm{d}\, m(S_A)$ is the measure of the non-compact symplectic group $Sp(2)$.

G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, *087901 (2004).*

Volume of Gaussian channels

- The range of coordinates that define a physical Gaussian state:

$$\begin{cases} 0 \leq \mu_{A/\sigma} \leq 1, \qquad \mu_A\mu_\sigma \leq \mu \leq \dfrac{\mu_A\mu_\sigma}{\mu_A\mu_\sigma + |\mu_A - \mu_\sigma|}, \\ \dfrac{2}{\mu} + \dfrac{(\mu_A - \mu_\sigma)^2}{\mu_A^2\mu_\sigma^2} \leq \Delta \leq \min\left\{ -\dfrac{2}{\mu} + \dfrac{(\mu_A + \mu_\sigma)^2}{\mu_A^2\mu_\sigma^2}, 1 + \dfrac{1}{\mu^2} \right\}. \end{cases} \tag{17}$$

- The volume of all one-mode Gaussian channels is equal to

$$V_{GC} = C \iiint_{\mathcal{CP}} \frac{\mu^{11/2}}{64\sqrt{2}\mu_A^3\mu_\sigma^2} \,\mathrm{d}\,\mu_A \,\mathrm{d}\,\mu \,\mathrm{d}\,\Delta = C \frac{4 + \mu_\sigma^{9/2}(9\mu_\sigma^2 - 13)}{18018\sqrt{2}\mu_\sigma^3}, \tag{18}$$

where $\mathcal{CP}$ is the region determined by conditions (17), and the infinite constant

$$C = \int_{\mathcal{M}} \mathrm{d}\, m(S_A) \int_0^{2\pi} \mathrm{d}\,\theta. \tag{19}$$

Volume of entanglement breaking channels

Definition 4

A quantum channel $\Lambda : \mathcal{H}_B \to \mathcal{H}_A$ is entanglement breaking if and only if

$$\rho_{AB} = (\Lambda \otimes \mathbb{I}_B)(\rho) \tag{20}$$

is separable for all states ρ.

For the Gaussian channels, it is enough that ρ_{AB} is separable for ρ_Ω with a marginal $\sigma = \mathrm{Tr}_A \, \rho_{AB}$.

Theorem 5

The two-mode Gaussian state ρ_{AB} is separable if and only if it satisfies the Peres-Horodecki criterion

$$\det(\Sigma_{PPT} + i\Omega) \geq 0, \tag{21}$$

where $\Sigma_{PPT} = \Theta\Sigma\Theta$ and $\Theta = \mathrm{diag}(-1,1,1,1)$.

R. Simon, Phys. Rev. Lett. **84***, 2726 (2000).*

Volume of entanglement breaking channels

- In the seralian-purity coordinates, condition $\det(\Sigma_{PPT} + i\Omega) \geq 0$ reads

$$1 + \frac{1}{\mu^2} + \Delta - \frac{2}{\mu_A^2} - \frac{2}{\mu_\sigma^2} \geq 0. \tag{22}$$

- The volume of all entanglement breaking one-mode Gaussian channels is equal to

$$\begin{aligned} V_{EBC} &= C \iiint_{\mathcal{SEP}} \frac{\mu^{11/2}}{64\sqrt{2}\mu_A^3 \mu_\sigma^2} \, \mathrm{d}\,\mu_A \, \mathrm{d}\,\mu \, \mathrm{d}\,\Delta \\ &= C \frac{\sqrt{\mu_\sigma}(1-\mu_\sigma)^2(11+9\mu_\sigma)}{18018\sqrt{2}}, \end{aligned} \tag{23}$$

where $\mathcal{SEP}$ is the physicality region $\mathcal{CP}$ with the additional constraint (22).

Incompatibility breaking channels

Definition 6

A quantum channel $\Lambda : \mathcal{H}_B \rightarrow \mathcal{H}_A$ is incompatibility breaking if and only if

$$\rho_{AB} = (\Lambda \otimes \mathbb{I}_B)(\rho) \tag{24}$$

is non-steerable for all states ρ.

Theorem 7

A one-mode Gaussian channel $\Lambda : \mathcal{H}_B \rightarrow \mathcal{H}_A$ is incompatibility breaking if and only if

$$\Sigma + i(0 \oplus \omega) \geq 0, \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{25}$$

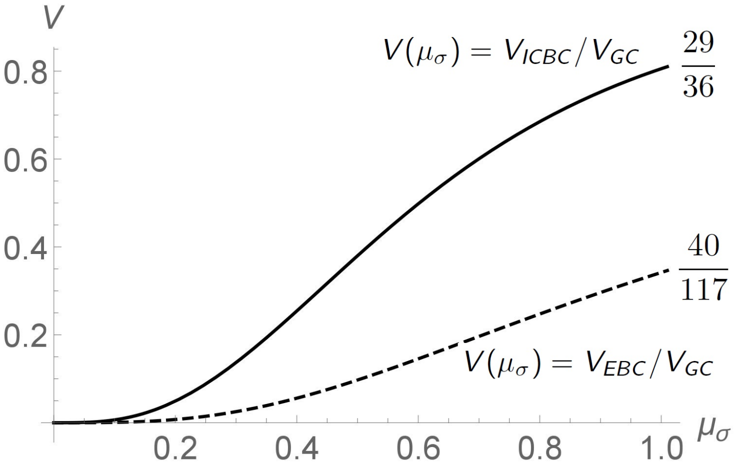
T. Heinosaari, J. Kiukas, and J. Schultz, J. Math. Phys. **56**, *082202 (2015).*

Incompatibility breaking channels

- In the purity-seralian coordinates, condition $\Sigma + i(0 \oplus \omega) \geq 0$ is equivalent to
$$\mu \leq \mu_A. \tag{26}$$
- The volume of all incompatibility breaking one-mode Gaussian channels is equal to
$$\begin{aligned} V_{ICBC} &= C \iiint_{\mathcal{NS}} \frac{\mu^{11/2}}{64\sqrt{2}\mu_A^3\mu_\sigma^2} \, \mathrm{d}\,\mu_A \, \mathrm{d}\,\mu \, \mathrm{d}\,\Delta \\ &= C \frac{\sqrt{\mu_\sigma}\left(-13\mu_\sigma + 9\mu_\sigma^3 - \frac{8\sqrt{2}(-11+7\mu_\sigma)}{(1+\mu_\sigma)^{7/2}}\right)}{18018\sqrt{2}}, \end{aligned} \tag{27}$$

where $\mathcal{NS}$ is the physicality region $\mathcal{CP}$ with the additional constraint (26).

Volume ratio



Partial knowledge about the system

Assume that our knowledge about a two-mode CJ Gaussian state is limited to the values of total μ and marginal $\mu_{A/\sigma}$ purities.

$$\mu_A\mu_\sigma \leq \mu \leq \frac{\mu_A\mu_\sigma}{\mu_A + \mu_\sigma - \mu_A\mu_\sigma} \qquad \text{(separable states)} \tag{28}$$

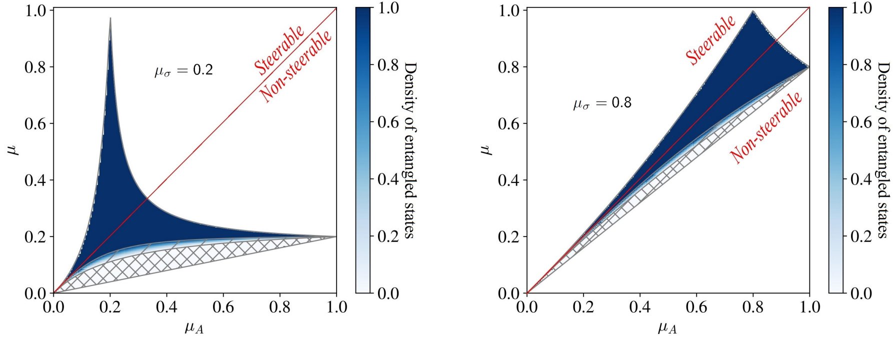
$$\frac{\mu_A\mu_\sigma}{\mu_A + \mu_\sigma - \mu_A\mu_\sigma} \leq \mu \leq \frac{\mu_A\mu_\sigma}{\sqrt{\mu_A^2 + \mu_\sigma^2 - \mu_A^2\mu_\sigma^2}} \qquad \text{(coexistence region)} \tag{29}$$

$$\frac{\mu_A\mu_\sigma}{\sqrt{\mu_A^2 + \mu_\sigma^2 - \mu_A^2\mu_\sigma^2}} \leq \mu \leq \frac{\mu_A\mu_\sigma}{\mu_A\mu_\sigma + |\mu_A - \mu_\sigma|} \qquad \text{(entangled states)} \tag{30}$$

In the coexistence region, it is impossible to distinguish between the separable and entangled states without the full knowledge about the system.

G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, *087901 (2004).*

Partial knowledge about the system



Summary

1. The geometrical properties of the Gaussian channels Λ can be determined by analyzing the properties of the corresponding Choi-Jamiołkowski states ρ_{AB}.
2. The volume element of the one-mode Gaussian channels depends on the total and marginal purities of ρ_{AB}, and it is flat in the seralian.
3. The volume ratios of the one-mode entanglement and incompatibility breaking Gaussian channels are monotonically increasing functions of μ_σ.
4. In many cases, it is possible to say whether a given one-mode Gaussian channel is entanglement or incompatibility breaking from the purities μ, $\mu_{A/\sigma}$ alone.

End notes

More on the topic soon on the ArXiv:

K. Siudzińska, K. Luoma, and W. T. Strunz, *On the geometry of one-mode Gaussian channels*.

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