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Quantum mutual information based on quantum conditional probability

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Plan of my talk

1. Motivation:
   -Simple review of the classical mutual entropy with three types of interpretation-

2. (1) Tow types of quantum conditional probability

   (2) Four scenarios of mutual entropy

   (3) Their relations

3. The measurement-induced disturbance of the quantum channel Φ

4. Summary

Classical information theory

Composite system $x \otimes y = \{(x_i, y_j)\}$ with its probabilities

$$p^{xy} = (p_{ij}^{xy}),\ \ p^x = (p_i^x),\ \ p^y = (p_j^y),\ \ p_{|i}^y = (p_{j|i})$$

Bayes formula : $p_{j|i} = \dfrac{p_{ij}^{xy}}{p_i^x} \Leftrightarrow p_{ij}^{xy} = p_i^x p_{j|i}$

$$S(x) = S(p^x) = -\sum_i p_i^x \log p_i^x$$

< Mutual entropy by relative entropy>

$$I(x,y) \equiv \sum p_{ij}^{xy} \log \frac{p_{ij}^{xy}}{p_i^x p_j^y} = S(x) + S(y) - S(x \otimes y) \quad (1)$$

$$= \sum p_i^x p_{j|i} \log \frac{p_{j|i}}{p_j^y} = S(y) - S(y|x) \quad (2)$$

Interpretation of mutual entropy:

(1) Correlation:
The common information encoded into x and y.

(2) Information gain by measurement:

$$S(y|x) \equiv \sum_i p_i^x S(p_{|i}^y) = \sum_i p_i^x \left(-\sum_j p_{j|i} \log p_{j|i}\right)$$

The conditional entropy measures how uncertain we are of y on the average when we know x.

< Mutual entropy on communication scheme >

Input system: $\left(x, p^x\right)$, Output system: $\left(y, p^y\right)$

Channel: $T = \left(T_{ij}\right) = \left(p_{j|i}\right)$ <u>transition prob. matrix</u>

$$p^y = T\left(p^x\right) \left(\leftrightarrow p_j^y = \sum_i p_i^x T_{ij}\right)$$

$$I(x, y) = \sum_{ij} p_i^x p_{j|i} \log \frac{p_{j|i}}{\sum_k p_k^x p_{j|k}} = I\left(p^x; T\right) \quad (3)$$

$$= S\left(p^x\right) + S\left(T\left(p^x\right)\right) - S\left(p_T^{xy}\right) \quad (1)'$$

$$= S\left(T\left(p^x\right)\right) - \sum_i p_i^x S\left(p_{|i}^y\right) \quad (2)'$$

(3) Transmitted information through the channel:

The amount received less the part of this which is due to noise.

<Shannon's fundamental inequality>

$$0 \le I\left(p^x; T\right) \le \min\left\{S\left(p^x\right), S\left(T\left(p^x\right)\right)\right\}$$

$I(x, y)$ has three types of interpretation at the same time based on Bayes formula

$$I(x, y) = S(x) + S(y) - S(x \otimes y)$$

$$= S\left(T\left(p^x\right)\right) - \sum_i p_i^x S\left(p_{|i}^y\right) = I\left(p^x; T\right)$$

How about quantum composite system?

Mutual entropies on a quantum composite system

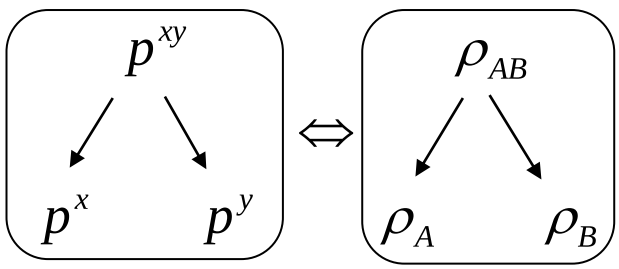
Composite system $\mathrm{H}_A \otimes \mathrm{H}_B$ with its density operators

$$\rho_{AB} \in \mathrm{S}(\mathrm{H}_{AB})$$

$$\mathrm{Tr}_B \rho_{AB} = \rho_A \in \mathrm{S}(\mathrm{H}_A),\ \ \mathrm{Tr}_A \rho_{AB} = \rho_B \in \mathrm{S}(\mathrm{H}_B)$$

$$S(\rho_A) = -\mathrm{tr}\rho_A \log \rho_A$$

<Quantum mutual entropy: scenario 1 > (by some authors)



$$I(A,B)_\rho = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

A measure of a quantum correlation encoded into a compound state ρ_{AB}

(Via quantum relative entropy)

$$D(\rho \| \sigma) \equiv \mathrm{Tr}\rho(\log \rho - \log \sigma)$$

(by Umegaki)

$$I(A,B)_\rho \equiv D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

$$= \mathrm{Tr}(\rho_{AB} \log \rho_{AB} - \log \rho_A \otimes \rho_B)$$

$$= S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

$I(A,B)_\rho$ **has a same structure with** $I(x,y)$**.**

<Quantum mutual entropy via quantum channel: scenario 2 >

The quantum channel Φ via Stinespring representation:

$$\Phi(\rho_A) = \mathrm{Tr}_{\mathrm{E}} V \rho V^*$$

where $V : \mathrm{H}_A \to \mathrm{H}_B \otimes \mathrm{E}$ is a linear isometry
and E may be interpreted as an environment Hilbert space.

$$\Downarrow$$

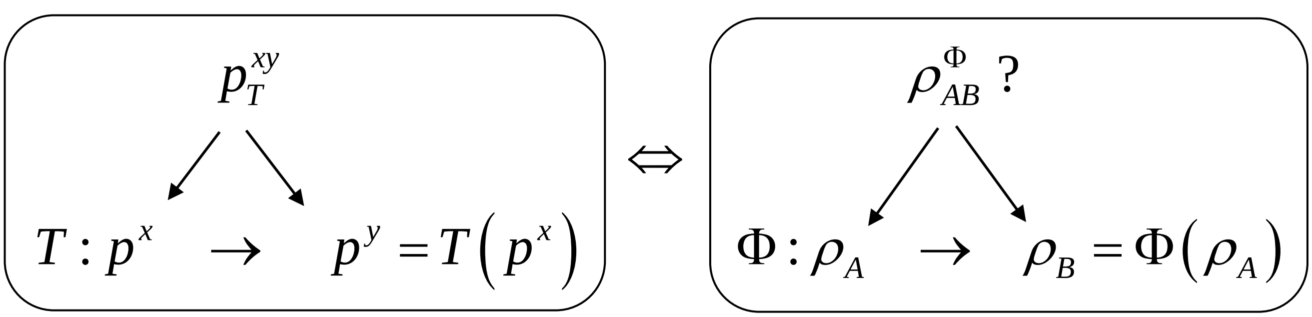
Quantum mutual information via channel Φ in the context of quantum data processing

Entropy exchange (by B. Schumacher):

$$S(\rho_A, \Phi) \equiv S(\rho_{\mathrm{E}})$$

where $\rho_{\mathrm{E}} = \Phi^{\mathrm{c}}(\rho_A) = \mathrm{Tr}_B V \rho_A V^*$. One can define

$$I(\rho_A ; \Phi) \equiv S(\rho_A) + S(\Phi(\rho_A)) - S(\rho_A, \Phi)$$



$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$I(x, y) = I(p^X ; T) \qquad\qquad I(A, B)_\rho \ ? \ I(\rho_A ; \Phi)$$

How can we connect scenario 1 and scenario 2?

Quantum conditional probability and quantum Bayse formula

Quantum conditional probability (by Cref and Adami)

$$\rho_{B|A} = \lim_{n\to\infty}\left[\rho_{AB}^{1/n}\left(\rho_A^{-1/n}\otimes I_B\right)\right]^n$$

then, we have

$$S(B|A)_\rho \triangleq S(\rho_{AB}) - S(\rho_A) = -\mathrm{Tr}\rho_{AB}\log\rho_{B|A}$$

In this talk:

another two types of quantum conditional probability (QCP for short)

< **QCP type 1**: full quantum (by Asorey et.al. or Leifer) >

For a given ρ_A and Φ,

QCP via Choi-Jamiolkowski isomorphism $\pi_{B|A}^{q-q}$:

$$\pi_{B|A}^{q-q} \equiv \sum_{i,j} E_{ij}\otimes\Phi(E_{ij})$$

Compound state ρ_{AB}^{q-q}:

$$\rho_{AB}^{q-q} = \left(\rho_A^{1/2}\otimes I_B\right)\pi_{B|A}^{q-q}\left(\rho_A^{1/2}\otimes I_B\right)$$

where $E_{ij} = |e_i\rangle\langle e_j|$ and $|e_i\rangle$ is the eigenbasis of ρ_A, i.e.,

$$\rho_A = \sum_i p_i |e_i\rangle\langle e_i| = \sum_i p_i\Pi_i$$

then,

$$(1)\quad \pi_{B|A}^{q-q} \geq 0 \qquad (2)\quad \mathrm{Tr}_B\pi_{B|A}^{q-q} = I_A$$

Quantum Bayse formula

$$\rho_{AB}^{q-q} = \sum_{i,j} \sqrt{p_i}\sqrt{p_j} E_{ij} \otimes \Phi\left(E_{ij}\right)$$

$$\Phi : \rho_A = \mathrm{Tr}_B \rho_{AB}^{q-q} \longrightarrow \rho_B = \mathrm{Tr}_A \rho_{AB}^{q-q} = \Phi\left(\rho_A\right)$$

Theorem 1

$$I\left(A, B\right)_\rho = I\left(\rho_A ; \Phi\right)$$

Proof：

$$I\left(A, B\right)_\rho = S\left(\rho_A\right) + S\left(\rho_B\right) - S\left(\rho_{AB}^{q-q}\right)$$

$$= S\left(\rho_A\right) + S\left(\Phi\left(\rho_A\right)\right) - S\left(\rho_{AB}^{q-q}\right)$$

we can show $S\left(\rho_{AB}^{q-q}\right) = S\left(\rho_A : \Phi\right)$, therefore

$$= S\left(\rho_A\right) + S\left(\Phi\left(\rho_A\right)\right) - S\left(\rho_A, \Phi\right)$$

$$= I\left(\rho_A ; \Phi\right)$$

The quantum version of the classical equality

$$I\left(x, y\right) = I\left(p^x ; T\right)$$

Notice:

In general $I\left(A, B\right)_\rho$ or $I\left(\rho_A ; \Phi\right)$ violates the Shannon's fundamental inequality, for example in the case of pure entangled state.

We rewrite $I\left(\rho_A, \Phi\right)$ by $I^{Q-Q}\left(\rho_A ; \Phi\right)$

< **QCP type 2**: semi quantum (by Ohya) >

For a given ρ_A and Φ

Compound state ρ_{AB}^{O} :

$$\rho_{AB}^{\mathrm{O}} \equiv \sum_i p_i E_{ii} \otimes \Phi(E_{ii}) = \sum_i p_i \Pi_i \otimes \Phi(\Pi_i) =: \rho_{AB}^{c-q}$$

$$\rho_A = \sum_i p_i |e_i\rangle\langle e_i| = \sum_i p_i \Pi_i$$

QCP $\pi_{B|A}^{c-q}$: $\pi_{B|A}^{c-q} \equiv \sum_i E_{ii} \otimes \Phi(E_{ii}) = \sum_i \Pi_i \otimes \Phi(\Pi_i)$

$$\Downarrow$$

$$\rho_{AB}^{c-q} = \left(\rho_A^{1/2} \otimes I_B\right) \pi_{B|A}^{c-q} \left(\rho_A^{1/2} \otimes I_B\right)$$

Proposition 1 The quantum analog of conditional entropy

$$-\mathrm{Tr}\rho_{AB}^{c-q} \log \pi_{B|A}^{c-q} = \sum_i p_i S\left(\rho_{B|i}\right) \quad (\Leftrightarrow \sum_i p_i^x S\left(p_{|i}^y\right))$$

where $\rho_{B|i} = \Phi(\Pi_i)$ ($\leftrightarrow$ conditional state).

The mutual entropy via Φ: scenario 3 (by Ohya)

$$I^{\mathrm{O}}(p_A;\Phi) \equiv \sup_{\Pi_i} D\left(\rho_{AB}^{c-q} \| \rho_A \otimes \Phi(\rho_A)\right) =: I^{C-Q}(\rho_A;\Phi)$$

where the sup is taken all decomposition $\rho_A = \sum_i p_i \Pi_i$.

Proposition 2

$$I^{C-Q}(\rho_A;\Phi) = S\left(\Phi(\rho_A)\right) - \inf_{\Pi_i} \sum_k p_k S\left(\Phi(\Pi_k)\right)$$

$$= \sup_{\Pi_i} \sum_k p_k D\left(\Phi(\Pi_k) \| \Phi(\rho_A)\right)$$

Theorem 2 $\quad 0 \le I^{C-Q}(p_A;\Phi) \le \min\left\{S(\rho_A), S\left(\Phi(\rho_A)\right)\right\}$

< The relation between $I(A,B)_{\rho}=I^{Q-Q}(\rho_A;\Phi)$

and $I^{C-Q}(p_A;\Phi)$**>**

Proposition 3

Let Π be the quantum channel defined by

$$\Pi(\rho)\equiv\sum_i\Pi_i\rho\Pi_i\,.$$

One has

$$\boxed{I^{C-Q}(\rho_A;\Phi)=\sup_{\Pi_i}I^{Q-Q}(\rho_A;\Phi_{\Pi})}$$

where $\Phi_{\Pi}=\Phi\circ\Pi$, and the sup is taken all decomposition $\rho_A=\sum_i p_i\Pi_i$.

Proposition 4 One has the following relation

$$D\left(\rho_{AB}^{q-q}\middle\|\rho_A\otimes\rho_B\right)=D\left(\rho_{AB}^{q-q}\middle\|\rho_{AB}^{c-q}\right)+D\left(\rho_{AB}^{c-q}\middle\|\rho_A\otimes\rho_B\right)$$

and hence

$$\boxed{I^{Q-Q}(\rho_A;\Phi)=\inf_{\Pi_i}D\left(\rho_{AB}^{q-q}\middle\|\rho_{AB}^{c-q}\right)+I^{C-Q}(\rho_A;\Phi)}$$

Note: the difference

$$Q(\rho_A;\Phi):=I^{Q-Q}(\rho_A;\Phi)-I^{C-Q}(\rho_A;\Phi)=\inf_{\Pi_i}D\left(\rho_{AB}^{q-q}\middle\|\rho_{AB}^{c-q}\right)$$

defines relative entropy distance between ρ_{AB}^{q-q} and the closest ρ_{AB}^{c-q} with the same marginal states ρ_A and $\rho_B=\Phi(\rho_A)=\Phi_{\Pi}(\rho_A)$.

The quantum discord vs $Q(\rho_A;\Phi)$

The notion of quantum discord:

(by Ollivier & Zurek or Henderson & Vedral)

Step1: a measurement on A part by the correction of one-dimensional projectors Π_k in H_A satisfying $\sum_k \Pi_k = I_A$,

$$\rho_{AB} \;\to\; \rho_{AB}^{C-Q} := [\Pi \otimes \mathrm{id}_B]\rho_{AB} = \sum_k (\Pi_k \otimes I_B)\rho_{AB}(\Pi_k \otimes I_B)$$

Step2: the state change after the measurement with outcome k

$$k \;\to\; \rho_{AB|k} \equiv \frac{1}{p_k}(\Pi_k \otimes I_B)\rho_{AB}(\Pi_k \otimes I_B) \;\to\; \rho_{B|k} = \mathrm{Tr}_A \rho_{AB|k}$$

where $p_k = \mathrm{Tr}(\Pi_k \otimes I_B)\rho_{AB}(\Pi_k \otimes I_B) = \mathrm{Tr}(\Pi_k \rho_A)$.

< Quantum discord - scenario 4 ->

$$\mathsf{D}(\rho_{AB}) \equiv I(A:B)_\rho - J(B|A)$$

where $J(B|A) \equiv \sup_{\{\Pi_k\}} I(\rho_{AB}|\{\Pi_k\})$ and

$$I(\rho_{AB}|\{\Pi_k\}) \equiv S(\rho_B) - \sum_k p_k S(\rho_{B|k}) = D(\rho_{AB}^{C-Q} \| \rho_A \otimes \rho_B)$$

where

$$\rho_{AB}^{C-Q} = \sum_k p_k \Pi_k \otimes \rho_{B|k}$$

with $\rho_A = \sum_k p_k \Pi_k$ and $\rho_B = \sum_k p_k \rho_{B|k}$

$\mathsf{D}(\rho_{AB})$ may be equivalently $Q(\rho_A;\Phi)$

Let replace ρ_{AB} by ρ_{AB}^{q-q}, then

$$\begin{aligned}\mathsf{D}\left(\rho_{AB}^{q-q}\right) &= \inf_{\Pi_i}\left\{I(A,B)_{\rho} - I(A,B)_{\rho^{c-q}}\right\} \\ &= \inf_{\Pi_i}\left\{I^{Q-Q}(\rho_A;\Phi) - D\left(\rho_{AB}^{c-q}\middle\|\rho_A\otimes\Phi(\rho_A)\right)\right\} \\ &= I^{Q-Q}(\rho_A;\Phi) - I^{C-Q}(\rho_A;\Phi) \\ &= Q(\rho_A;\Phi)\end{aligned}$$

Notice:

In the case of ρ_{AB}^{q-q} the rank-1 projectors have to commute with ρ_A.

Such quantity may be called the "**measurement-induced disturbance**" (by Luo)

= One performs a projective measurement on the A subsystem without perturbing its system. =

Definition 1 The measurement-induced disturbance of the channel Φ is defined by

$$\Delta(\Phi) \equiv \inf_{\rho_A} Q(\rho_A;\Phi)$$

$\Delta(\Phi)$ measures the distortion of Φ from the c-q channel Φ^{c-q} due to the fact:

$$\Delta\left(\Phi^{c-q}\right) = 0$$

Summary

Introducing the quantum analog of Bayse formula via QCP (full quantum and semi quantum) , we made clear the relation between several types of mutual entropy.

Type 1 (full quantum QCP $\pi_{B|A}^{q-q}$)

$$\rho_{AB}^{q-q} = \left(\rho_A^{1/2} \otimes I_B\right)\left(\sum_i E_{ij} \otimes \Phi\left(E_{ij}\right)\right)\left(\rho_A^{1/2} \otimes I_B\right)$$

we have

$$\begin{aligned} I(A, B)_\rho &= S(\rho_A) + S(\rho_B) - S\left(\rho_{AB}^{q-q}\right) \\ &= S(\rho_A) + S\left(\Phi(\rho_A)\right) - S(\rho_A, \Phi) = I^{Q-Q}(\rho_A; \Phi) \end{aligned}$$

Type 2 (semi quantum QCP $\pi_{B|A}^{c-q}$)

$$\rho_{AB}^{c-q} = \left(\rho_A^{1/2} \otimes I_B\right)\left(\sum_i \Pi_i \otimes \Phi\left(\Pi_i\right)\right)\left(\rho_A^{1/2} \otimes I_B\right)$$

we have

$$I^{C-Q}(\rho_A; \Phi) = S\left(\Phi(\rho_A)\right) - \inf_{\Pi_i} \sum_k p_k S\left(\Phi(\Pi_k)\right)$$

$$0 \le I^{C-Q}(p_A; \Phi) \le \min\left\{S(\rho_A), S\left(\Phi(\rho_A)\right)\right\}$$

$$I^{C-Q}(\rho_A; \Phi) = \sup_{\Pi_i} I^{Q-Q}(\rho_A; \Phi_\Pi) \quad (\Phi_\Pi = \Phi \circ \Pi)$$

Two types of QCP define
the measurement-induced disturbance of the channel Φ

$$Q(\rho_A; \Phi) := I^{Q-Q}(\rho_A; \Phi) - I^{C-Q}(\rho_A; \Phi) = \inf_{\Pi_i} D\left(\rho_{AB}^{q-q} \,\middle\|\, \rho_{AB}^{c-q}\right)$$

$$\Delta(\Phi) = \inf_{\rho_A} Q(\rho_A; \Phi)$$

Thank you for your attention!!