Operational relevance of resource theories of quantum measurements

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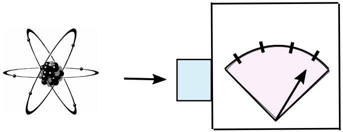
Introduction to Positive Operator Value Measures (POVMs)

- **M**:=[M_1 M_2 ..... M_n]. Each M_i are called **effect** of **M**.
- $\forall i \quad M_i \geq 0 \quad \sum_i M_i = \mathbb{I}$
- Set of all POVM in dimension d with n outcomes $:= \mathcal{P}(d, n)$



ρ

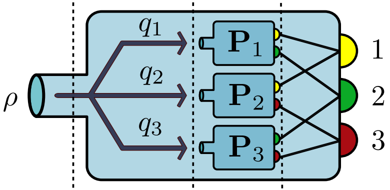
$\{M_i\}_{i=1}^{N}$

Born rule

$$p(i|\rho) = \mathrm{tr}(M_i \rho)$$

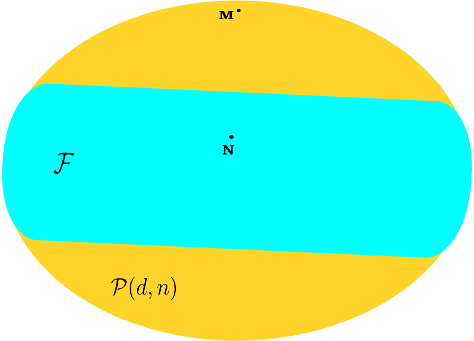
Introduction to Positive Operator Value Measures (POVMs)

- **Projective POVM**$:= \forall i,j \;\; M_i M_j = M_i \delta_{ij}$
- **Examples :** For $d = 2$; $M_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$ and $M_2 = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|-\rangle\langle -|$



- Set of POVM $\mathcal{P}(d,n)$ is **convex**. For $\lambda \in [0,1]$ convex combination of two POVM $\mathbf{M}, \mathbf{N} \in \mathcal{P}(d,n)$, is defined as $\lambda \mathbf{M} + (1-\lambda)\mathbf{N} \in \mathcal{P}(d,n)$

Free POVM and Resource POVMs

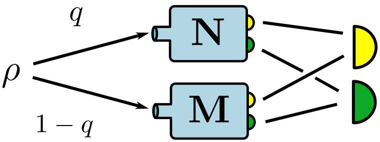


- **Free POVMs:** Any POVM $\mathbf{N} \in \mathcal{F}$. $\mathcal{F}$ is convex and closed.
- **Free Operation:** $\mathcal{Q} : \mathcal{P}(d,n) \to \mathcal{P}(d,n)$ such that $\mathcal{Q}(\mathcal{F}) \subseteq \mathcal{F}$
- **Resource POVMs:** Any POVM which is not free.

Resource theory of Informativeness

- **Free POVM**: $\mathbf{M} = \{\alpha_i \mathbb{I}\}_{i=1}^{n}$ with $\forall i \; \alpha_i \geq 0$ and $\sum_i^n \alpha_i = 1$
- **Free Operation**: Classical post processing. $\mathbf{M'} = \mathcal{Q}\mathbf{M}$ where $\mathcal{Q}$ is a stochastic matrix.
- **Resource POVM**: POVM whose effect are not proportional to Identity.

Skrzypczyk, Linden PRL 122, 140403 (2019)

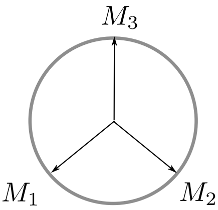


Given **N** and **M** are POVM we can have two classical manipulations on them

- **Randomisation** : The mixture of two projective measurement $q\mathbf{N} + (1-q)\mathbf{M}$ with effect $[q\mathbf{N} + (1-q)\mathbf{M}]_i = q\mathbf{N}_i + (1-q)\mathbf{M}_i$
- **Classical post-processing** : We can apply a stochastic matrix $\mathcal{Q} = q(i|j)$ on a POVM **M** which gives a POVM $[\mathcal{Q}(\mathbf{M}])_i = \Sigma_j q(i|j) M_j$

Resource theory of Non-projective simulability

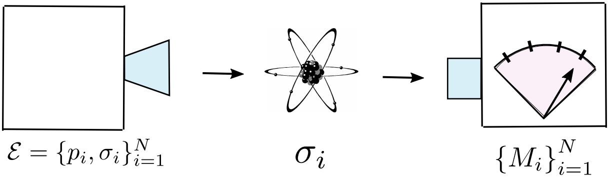
- **Free POVM**: Projective simulable measurement which can be described as randomisation and classical post-processing of projective measurement.
- **Free Operation**: Classical post-processing
- **Resource POVM** : POVM which are not projective simulable. For example $\mathcal{P}(d=2, n=3)$ trine POVM.



$$M_i = \frac{2}{3} |\psi_i\rangle\langle\psi_i|$$

Oszmaniec et. al Phys. Rev. Lett. 119 (19), 190501

Quantum state discrimination problem



- $\mathcal{E} := \{p_i, \sigma_i\}_{i=1}^k$ is a quantum source which produce quantum state σ_i with probability p_i.
- Given a POVM $\mathbf{M} \in \mathcal{P}(d, n)$, the success probability of discriminating quantum state σ_i from that source

$$\mathcal{P}_{succ}(\mathcal{E}, \mathbf{M}) = \sum_{i=1}^k p_i Tr(M_i \sigma_i)$$

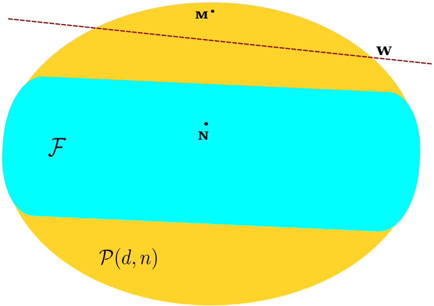
Witnessing a resource POVM via quantum state discrimination

Theorem

Given any *resource* POVM $\mathbf{M} \notin \mathcal{F}(d,n)$ $\exists$ $\mathcal{A} := \{p_i, \sigma_i\}$ such that

$$\frac{\mathcal{P}_{succ}(\mathcal{A},\mathbf{M})}{\max_{\mathbf{N}\in\mathcal{F}(d,n)}\mathcal{P}_{succ}(\mathcal{A},\mathbf{N})} > 1$$

Proof sketch:



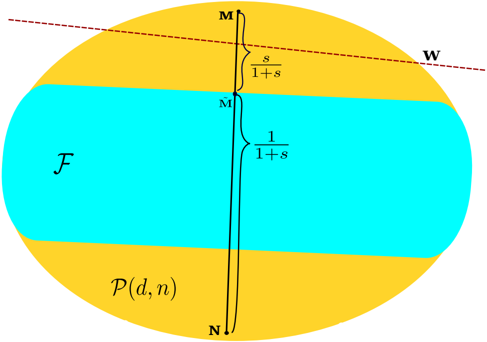
Witnessing a resource POVM via quantum state discrimination

- $\mathcal{W} := [W_1 \; W_2 \; .... \; W_n]$ such that $\sum_i Tr(W_i M_i) > 0$ and $\sum_i Tr(W_i N_i) \leq 0 \; \forall \; \mathbf{N} \in \mathcal{F}(d, n)$
- $\forall i \;\; W_i$ is Hermitian operator.
- $\tilde{W}_i := W_i + |\lambda_i| \mathbb{I}, \; \forall \; i$ where λ_i is the minimum eigenvalue W_i.
- $\mathcal{A} := \left\{ \frac{Tr(\tilde{W}_i)}{\sum_i Tr(\tilde{W}_i)}, \frac{\tilde{W}_i}{Tr(\tilde{W}_i)} \right\}$

Robustness of a POVM

- $\mathcal{R}_{\mathcal{F}}\colon\mathcal{P}(d,n) \to \mathbb{R}_+$ such that

$$\mathcal{R}_{\mathcal{F}}(\mathbf{M}) := \min_{\mathbf{N}\in\mathcal{P}(d,n)} \left\{ s \middle| \frac{\mathbf{M}+s\mathbf{N}}{1+s} \in \mathcal{F}(d,n) \right\}$$



Properties of robustness

- **Faithfulness:** $\mathcal{R}_{\mathcal{F}}(\mathbf{M}) = 0$ iff $\mathbf{M} \in \mathcal{F}$
- **Convexity:** $\mathcal{R}_{\mathcal{F}}(\lambda \mathbf{M}_1 + (1-\lambda)\mathbf{M}_2)) \leq \lambda \mathcal{R}_{\mathcal{F}}(\mathbf{M}_1) + (1-\lambda)\mathcal{R}_{\mathcal{F}}(\mathbf{M}_2)$ for $\lambda \in [0,1]$
- **Non increasing under any free operation $\mathcal{Q}$:** $\mathcal{R}_{\mathcal{F}}(\mathcal{Q}(\mathbf{M})) \leq \mathcal{R}_{\mathcal{F}}(\mathbf{M})$

Operational relevance of robustness quantum state discrimination

Theorem

$$\max_{\mathcal{A}} \frac{\mathcal{P}_{succ}(\mathcal{A},\mathbf{M})}{\max_{\mathbf{N}\in\mathcal{F}(d,n)}\mathcal{P}_{succ}(\mathcal{A},\mathbf{N})} = 1 + \mathcal{R}_{\mathcal{F}}(\mathbf{M})$$

Proof sketch: **LHS** $\leq$ **RHS**

- $\mathbf{M} + \mathcal{R}_{\mathcal{F}}(\mathbf{M})\mathbf{N} = (1 + \mathcal{R}_{\mathcal{F}}(\mathbf{M}))\mathbf{N}'$ for some $\mathbf{N}' \in \mathcal{F}(d, n)$
- $\mathcal{P}_{succ}(\mathcal{A}, \mathbf{M}) \leq (1 + \mathcal{R}_{\mathcal{F}}(\mathbf{M}))\mathcal{P}_{succ}(\mathcal{A}, \mathbf{N}') \leq (1 + \mathcal{R}_{\mathcal{F}}(\mathbf{M})) \max_{\mathbf{N}\in\mathcal{F}(d,n)} \mathcal{P}_{succ}(\mathcal{A}, \mathbf{N})$

Operational relevance of robustness quantum state discrimination

Proof Sketch : **LHS** $\geq$ **RHS**

- Find one ensemble for which equality holds.
- $$\begin{aligned} \text{maximize} \quad & \sum_i Tr(Z_i M_i) - 1 \\ \text{subject to} \quad & Z_i \geq 0, \; i = 1, \ldots, n. \\ & \sum_i Tr(Z_i N_i) \leq 1 \; \forall \mathbf{N} \in \mathcal{F} \end{aligned}$$
- **Strong Duality:** From the Slater's condition, this point $\mathbf{Z} = (\frac{\lambda \mathbb{I}}{d}, \ldots, \frac{\lambda \mathbb{I}}{d})$ strictly satisfies the constraints .

Incoherent measurement

- POVM $\mathbf{S}$ with effect $S_a = \sum_{i=1}^{d} q(a|i)|i\rangle\langle i|$
- $\mathcal{R}_{\mathcal{IC}}(\mathbf{M}) :=$

$$
\begin{aligned}
\text{minimize} \quad & s \\
\text{s.t} \quad & \frac{M_a + sN_a}{1+s} = \sum_i q(a|i)|i\rangle\langle i| \quad \forall a \\
& N_a \geq 0, \; a = 1, \ldots, n. \; \sum_a N_a = \mathbb{I} \\
& q(a|i) \geq 0, \; a = 1, \ldots, n. \; \sum_a q(a|i) = 1
\end{aligned}
$$

Incoherent measurement

- Dual characterization of the problem of robustness
- $\mathcal{R}_{\mathcal{IC}}(\mathbf{M}) :=$

$$\begin{aligned} \text{maximize} \quad & \sum_{a=1}^{n} Tr(Z_a M_a) - 1 \\ \text{subject to} \quad & \forall i, a, \langle i|Z_a|i\rangle = \langle i|Z_n|i\rangle \\ & Tr(Z_n) = 1 \end{aligned}$$

- $\mathcal{R}_{\mathcal{IC}}(\mathbf{M})$ is upper bounded by $d-1$
- Maximal $\mathcal{R}_{\mathcal{IC}}(\mathbf{M})$ is achieved for the measurement given by $|\tilde{j}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{i2\pi jk}{d}} |k\rangle$
- For POVM with $n \geq d$, Maximal $\mathcal{R}_{\mathcal{IC}}(\mathbf{M})$ is achieved **only** for the measurement given by $|\tilde{j}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i\theta_k} |k\rangle$

Separable measurement

- POVM **F** is called separable whose effects are separable $F_i = \sum_k Q_i^k \otimes \tilde{Q}_i^k$.
- Easy to characterize the superset of **LOCC** measurement.
- What is the maximal robustness (advantage in quantum state discrimination) that general measurements offer over separable measurements
- For bipartite case in $\mathbb{C}^d \otimes \mathbb{C}^d$

$$\min\{d_A, d_B\} - 1 \leq \max_{\mathbf{M}} \mathcal{R}_{Sep(AB)}(\mathbf{M}) \leq \min\{d_A, d_B\}$$

- For multi-partite case in $(\mathbb{C}^2)^{\otimes N}$

$$c\frac{2^N}{8N^2} \leq \max_{\mathbf{M}} R_{(N)}(\mathbf{M}) \leq 2^{\frac{3}{2}N-1} - 1$$

Conclusions and outlook

- Quantum state discrimination as a witnessing task for showing resource POVM advantageous over free POVM.
- Introducing robustness as a quantifier to characterize the gap between free and resource POVM.
- This result will hold for any resource theory whose free state is convex and closed.
- Operational interpretation in terms of other noise model like "white noise"
- Charecterizing robustness for projective simulable measurement and separable measurement.

Oszmaniec, Biswas Quantum 3, 133 (2019)

Operational relevance of robustness quantum state discrimination

- $$\begin{aligned} \text{maximize} \quad & \sum_i Tr(Z_i M_i) - 1 \\ \text{subject to} \quad & Z_i \geq 0, \ i = 1, \ldots, n. \\ & \sum_i Tr(Z_i N_i) \leq 1 \ \forall \mathbf{N} \in \mathcal{F} \end{aligned}$$

- $$\begin{aligned} \text{-minimize} \quad & \sum_i Tr(X_i M_i) + 1 \\ \text{subject to} \quad & X_i \leq 0, \ i = 1, \ldots, n. \\ & \sum_i Tr(X_i N_i) \geq 1 \ \forall \mathbf{N} \in \mathcal{F} \end{aligned}$$

Operational relevance of robustness quantum state discrimination

- $$\begin{aligned} \text{-minimize} \quad & \sum_i Tr(\tilde{X}_i M_i) \\ \text{subject to} \quad & \tilde{X}_i \leq \frac{\mathbb{I}}{d}, \ i = 1, \ldots, n. \\ & \sum_i Tr(\tilde{X}_i N_i) \geq 0 \ \forall \mathbf{N} \in \mathcal{F} \end{aligned}$$

- The corresponding **Lagrangian** of the problem with Lagrange multiplier $\mathbf{G} = (G_1, \ldots, G_n) \geq 0$ and $k(\mathbf{N}) \geq 0$

$$\mathcal{L}(\tilde{X}, \mathbf{G}, \{k(\mathbf{N})\}) = \sum_{i=1}^{n} \Bigg(Tr(\tilde{X}_i M_i) + Tr(\tilde{X}_i - \frac{\mathbb{I}}{d}) G_i - \int d\mathbf{N} k(\mathbf{N}) Tr(N_i X_i) \Bigg) ,$$

Operational relevance of robustness quantum state discrimination

- $$\begin{aligned} \text{minimize} \quad & \frac{1}{d}\sum_{i=1}^{n}(G_i) \\ \text{subject to} \quad & \mathbf{G}+\mathbf{M} = \int k(\mathbf{N})d\mathbf{N} \ , \\ & G_i \geq 0 \ \forall i \ , \\ & k(\mathbf{N}) \geq 0 \quad \forall \mathbf{N} \in \mathcal{F} \ . \end{aligned}$$
- This problem is equivalent to the $\mathcal{R}_{\mathcal{F}}(\mathbf{M})$

Strong Duality: From the Slater's condition, this point $\mathbf{Z} = (\frac{\lambda\mathbb{I}}{d}, \dots, \frac{\lambda\mathbb{I}}{d})$ strictly satisfies the constraints .