Discrimination of quantum orthogonal measurements of von Neumann

Łukasz Pawela, Zbigniew Puchała, Aleksandra Krawiec, Ryszard Kukulski, Michał Oszmaniec

IITiS

Institute of Theoretical and Applied Informatics, Polish Academy of Sciences

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Total variational distance

Calssical result

Let $\vec{p}, \vec{q}$ be discrete probability distributions on Γ. We choose one distribution with probability $\frac{1}{2}$ and sample form it.
Optimal probability of correct guessing of the true distribution after observing one sample is

$$p_{\mathrm{opt}} = \frac{1}{2} + \frac{1}{4}\|\vec{p} - \vec{q}\|_1 = \frac{1}{2} + \frac{1}{2}\max_{\Delta \subseteq \Gamma}\left(\sum_{i \in \Delta} p_i - q_i\right). \tag{1}$$

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This holds for quantum states!

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Maximum likelihood estimation

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Solution

- If we obtain H we say the coin is biased
- The probability of correct discrimination is $\frac{1}{2} + \frac{1}{4} \cdot \frac{2}{4} = \frac{5}{8}$

Quantum measurements

Definition

Measurement (POVM) $\mathcal{M}$ on a complex Euclidean space X is a collection of non-negative operators $\{M_i\}_{i\in\Gamma}$ such that

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Application of measurement

When a measurement is applied to a register whose state is ρ, an element of Γ is randomly selected as an outcome, with distribution

$$p_i = \mathrm{tr} M_i \rho. \tag{3}$$

Quantum channel

Measurement application as quantum channel

I will identify application of measurement with a quantum–to–classical channel, defined as

$$\mathcal{M} : \rho \mapsto \mathrm{diag}(\{\mathrm{tr} M_i \rho\}_{i \in \Gamma}). \quad (4)$$

Numerical range

Definition

$$W(A) = \{z \in \mathbb{C} : z = \langle\psi|A|\psi\rangle,\ |\psi\rangle \in \mathbb{C}^d,\ \langle\psi|\psi\rangle = 1\}.$$

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Some properties

- $W(A)$ is convex
- If $A^\dagger A = AA^\dagger$ then $W(A) = \mathrm{conv}(\lambda(A))$

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Example

Let $\mathcal{M} = \{M_i\}, \mathcal{N} = \{N_i\}$. If $\mathrm{tr} M_i N_i = 0$, then if we choose $\rho \simeq M_i$ and observe an output i, then we know for sure that $\mathcal{X} = \mathcal{M}$.

Probability of correct guessing

Calssical result

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Probability of correct guessing

In the case of quantum measurements we can optimize over input states and get

Probability of correct guessing

Optimal distinguishability using only disentangled input states, reads

$$p_{\mathrm{opt}} = \frac{1}{2} + \frac{1}{2} \max_{\Delta \subseteq \Gamma} \left\| \sum_{i \in \Delta} M_i - N_i \right\|_{\infty}. \tag{6}$$

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von Neumann measurements

Denote

- $\mathcal{P}_U = \{U|i\rangle\langle i|U^\dagger\}_i$,
- $\mathcal{P}_{\mathbf{1}} = \{|i\rangle\langle i|\}_i$

Entangled input states

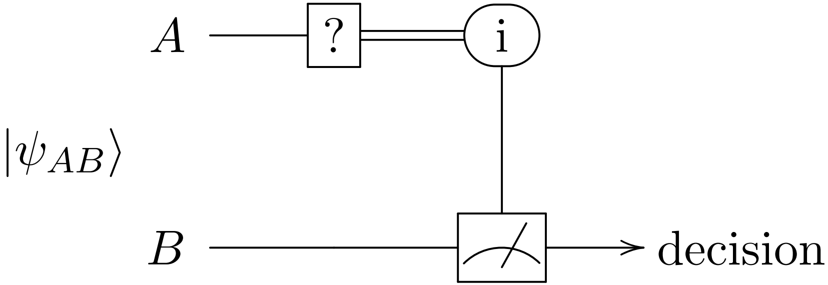

Figure: Entanglement-assisted discrimination

Entanglement assisted discrimination

Probability of correct distinction

$$p_{\text{opt}} = \frac{1}{2} + \frac{1}{4}\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond. \tag{7}$$

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Definition

Consider a linear Hermiticity-preserving mapping Φ. Completely bounded trace norm, also known as the diamond norm is defined as

$$\|\Phi\|_\diamond = \max_{|\psi\rangle\langle\psi|} \| (\Phi \otimes \mathbb{1}) (|\psi\rangle\langle\psi|)\|_1. \tag{8}$$

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Amount of entanglement needed

The Schmidt rank of the optimal input state is the dimensionality of the additional system needed for discrimination.

Discrimination of unitary channels

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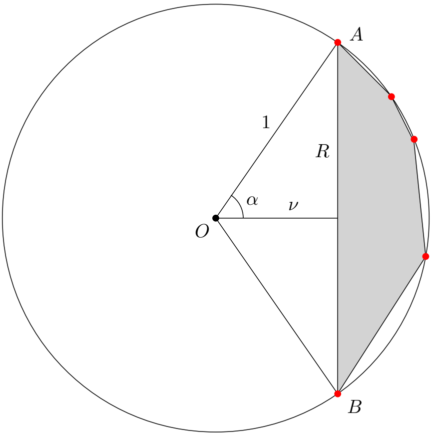
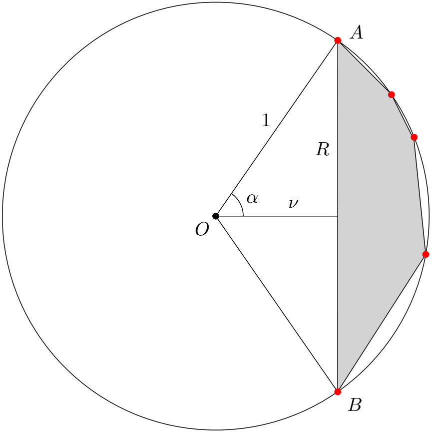
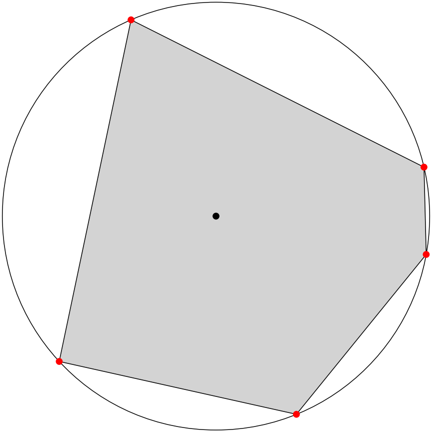


$$\|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond = 2\sqrt{1-\nu^2}$$

Discrimination of unitary channels



$$\left\|\Phi_U - \Phi_{\mathbb{1}}\right\|_\diamond = 2\sqrt{1 - \nu^2}$$



$$\left\|\Phi_U - \Phi_{\mathbb{1}}\right\|_\diamond = 2$$

Result for von Neumann POVMs

Relationship with unitary channel discrimination

Let $U, V \in \mathcal{U}_d$ and let $\mathcal{P}_U$ and $\mathcal{P}_V$ be two projective measurements. Let also $\mathcal{DU}_d$ be the set of diagonal unitary matrices of dimension d. Then

$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = \min_{E \in \mathcal{DU}_d} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond, \tag{9}$$

where Φ_U is unitary channel.

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Solution: convex program

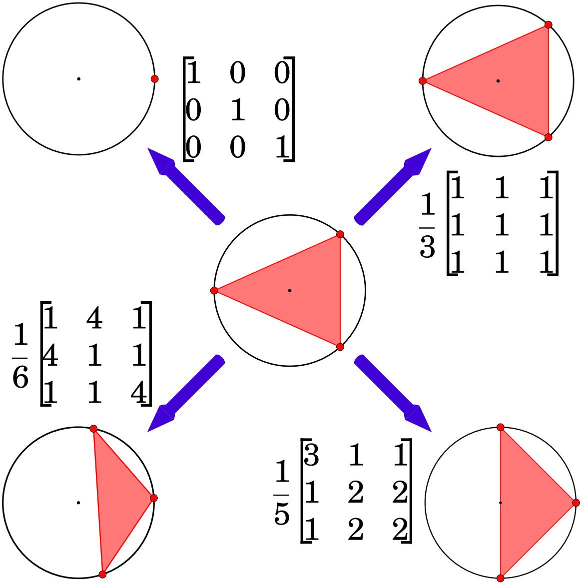
Convex problem

$$\begin{aligned} \text{minimize:} \quad & \sum_i |\langle i|\rho U|i\rangle| \\ \text{subject to:} \quad & \operatorname{tr}\rho = 1, \\ & \rho \geq 0. \end{aligned}$$

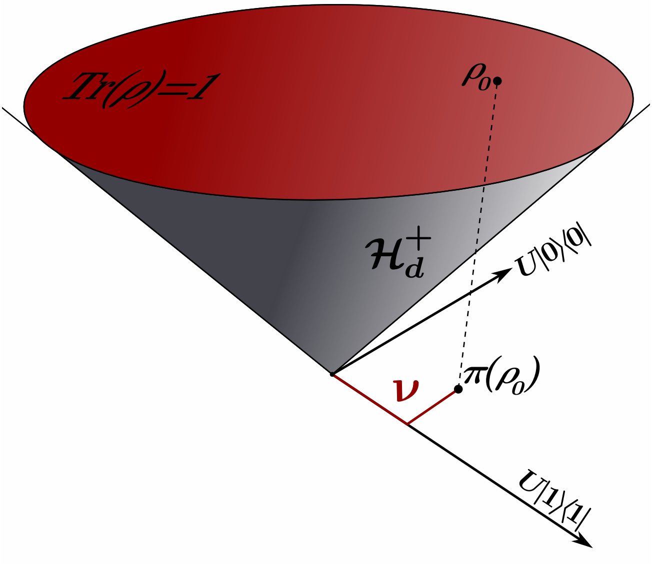
The minimum value ν of the above problem gives us a value of diamond norm

$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = 2\sqrt{1-\nu^2}. \tag{10}$$

Eigenvector dependence



Geometric interpretation of optimization



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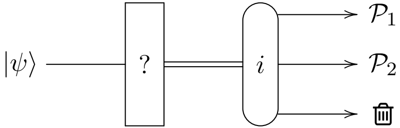

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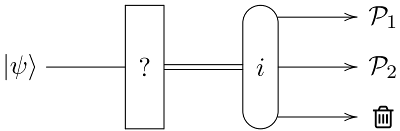

$$\tilde{p}_{\mathrm{u}}\left(\mathcal{P}_{\mathbf{1}}, \mathcal{P}_U\right) = \begin{cases} 1 & \text{if} \quad |U_{12}|^2 = 1 \\ \frac{1}{2}|U_{1,2}|^2 & \text{if} \quad |U_{12}|^2 < 1 \end{cases} \tag{11}$$

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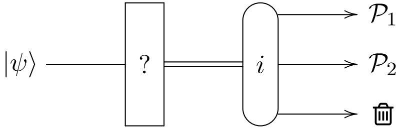

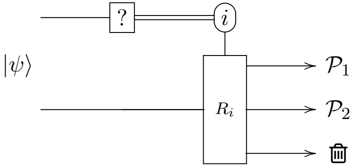

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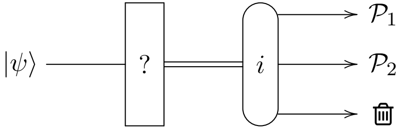

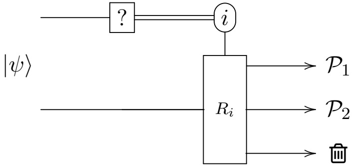

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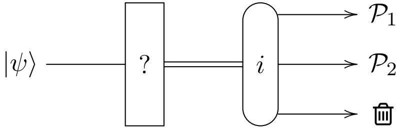

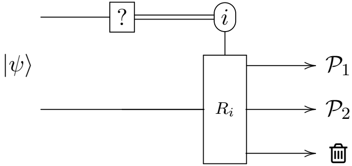

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Qubit case

$$p_{\mathrm{u}} = 1 - |U_{1,1}| \tag{13}$$

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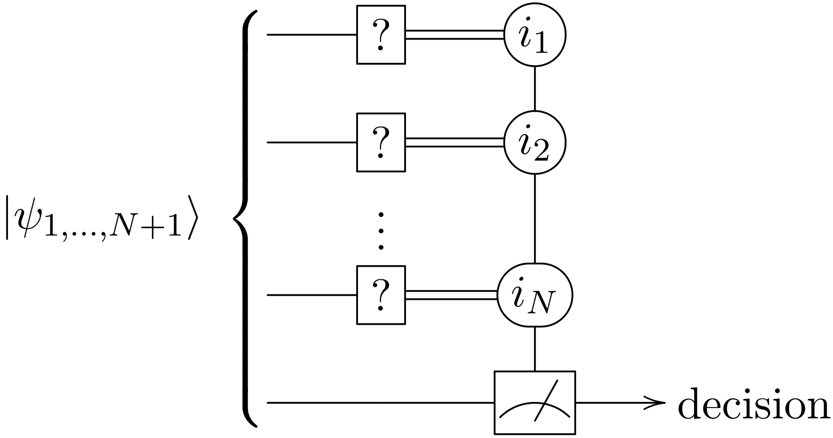
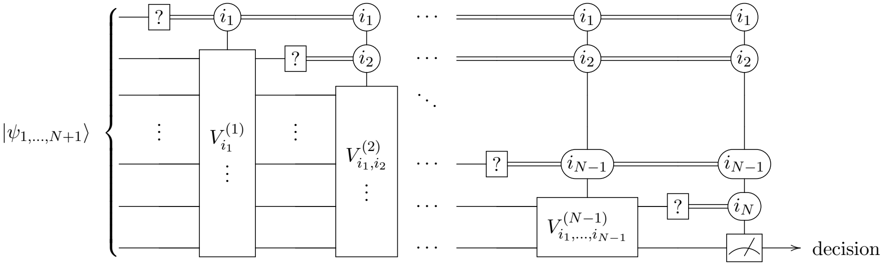
Parallel scenario



Adaptive scenario



Main result

Parallel scheme is optimal

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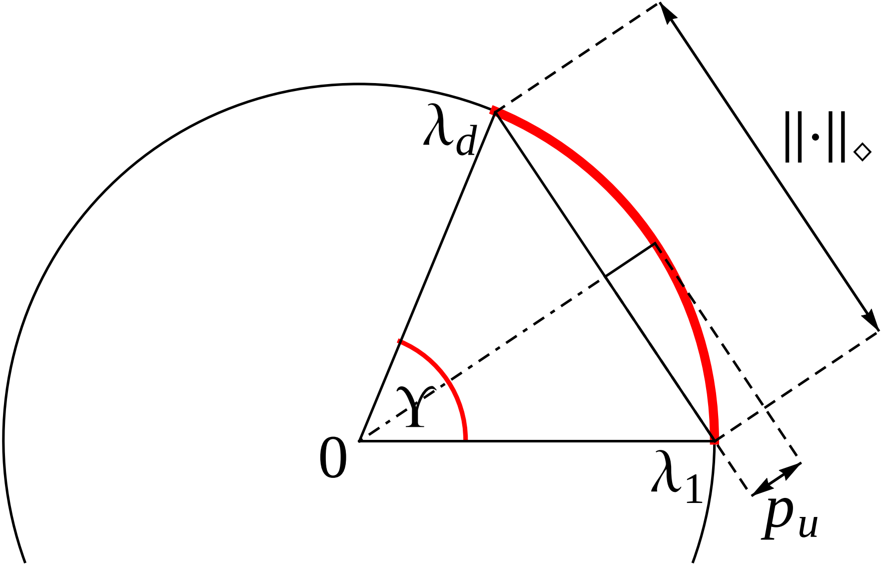
$$\begin{aligned} \nu &= \min_{\rho} \sum_i |\langle i|\rho U^{\otimes N}|i\rangle| \\ \|\cdot\|_\diamond = 2\sqrt{1-\nu^2} \, p &= \frac{1}{2} + \frac{\sqrt{1-\nu^2}}{2} \\ p_{\mathrm{u}} &= 1 - \nu \end{aligned} \tag{14}$$

Geometric interpretation



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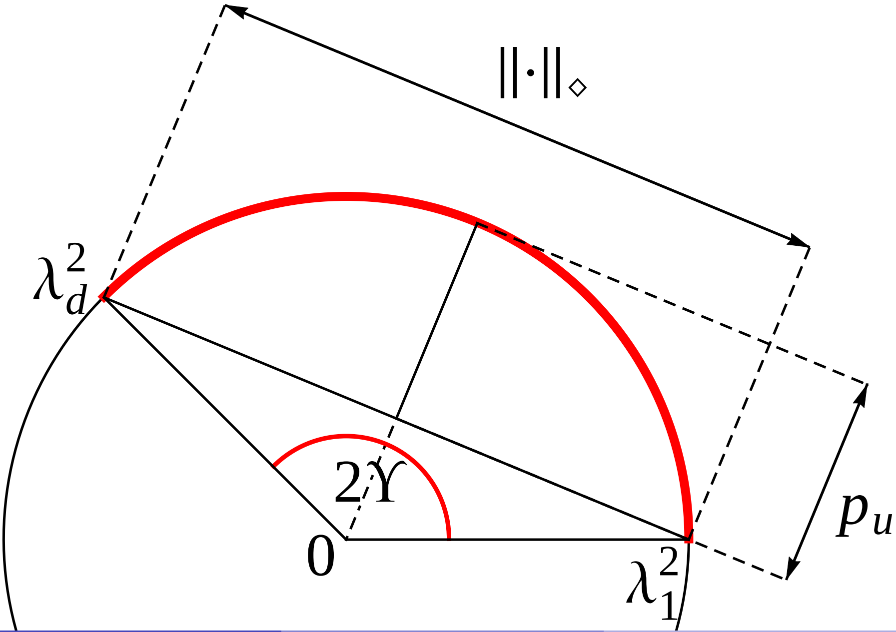

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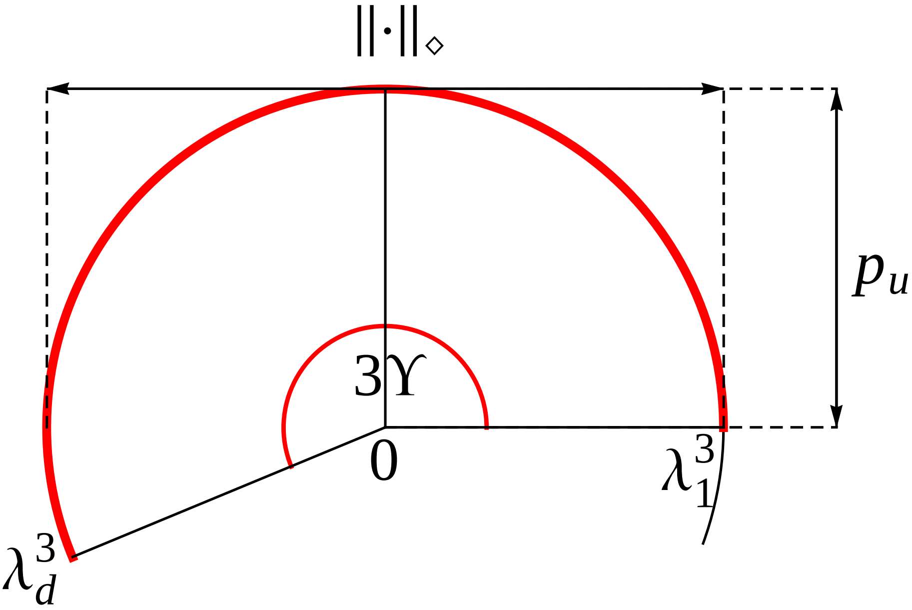

Based on

- Z. Puchała, Ł. Pawela, A Krawiec, R. Kukulski and M. Oszmaniec, *Multiple-shot and unambiguous discrimination of von Neumann measurements*, arXiv:1810:05122
- Z. Puchała, Ł. Pawela, A Krawiec and R. Kukulski *Strategies for optimal single-shot discrimination of quantum measurements*, PRA 98(4), arXiv:1804.05856

Thank you for your attention!