Data-Driven Inference and Observationally Complete Devices

joint work with: M. Dall'Arno, A. Bisio, A. Tosini

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Toruń, Poland, 16 June 2019

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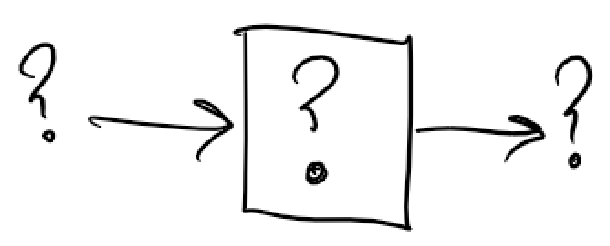

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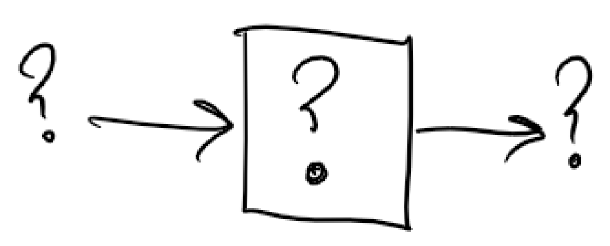

how can we infer anything about it?

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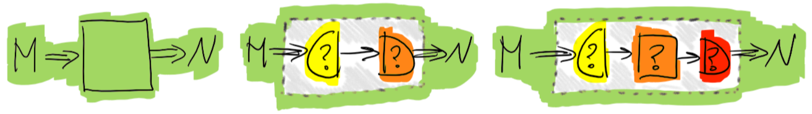

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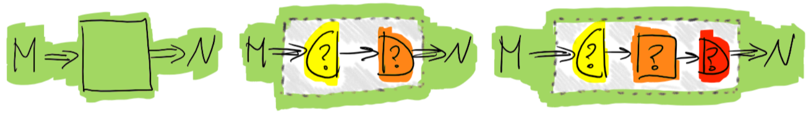

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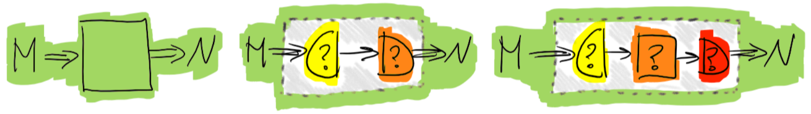

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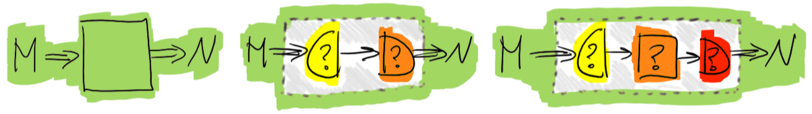

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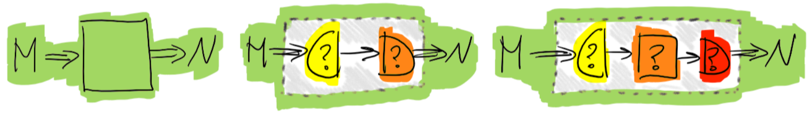

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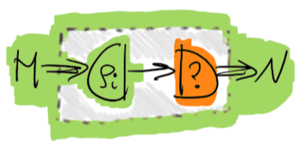
- case-study in this talk: measurement inference

Conventional tomography



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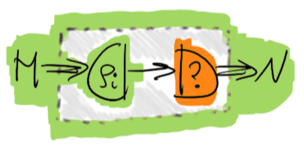

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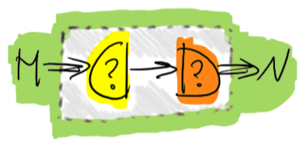

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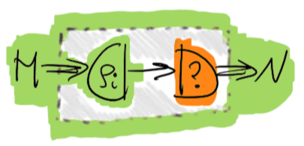

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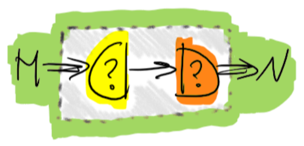

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Motivation: to break (or at least to loosen) the circular argument on which conventional tomography relies

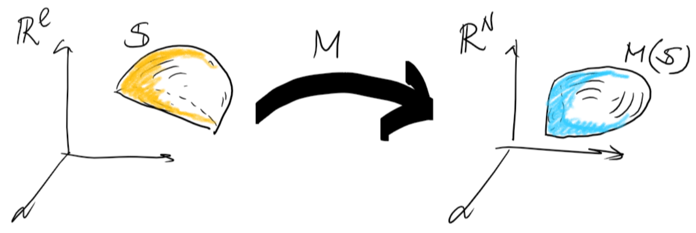
Wigner's Other Chain

As Wigner put it:

> *[...] the experimentalist uses certain apparatus to measure the position, let us say, or the momentum, or the angular momentum. Now, how does the experimentalist know that this apparatus will measure for him the position? "Oh," you say, "he observed the apparatus. He looked at it." Well that means that he carried out a measurement on it. How did he know that the apparatus with which he carried out that measurement will tell him the properties of the apparatus? Fundamentally, this is again a chain which has no beginning.* **And at the end we have to say, "We learned that as children how to judge what is around us.**"

[E.P. Wigner, Lecture at the Conference on the Foundations of Quantum Mechanics, Xavier University, Cincinnati, 1962.]

Measurement Representation



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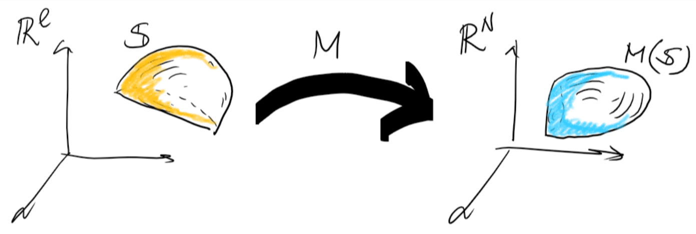

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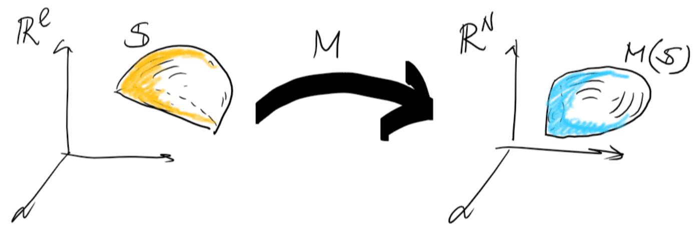

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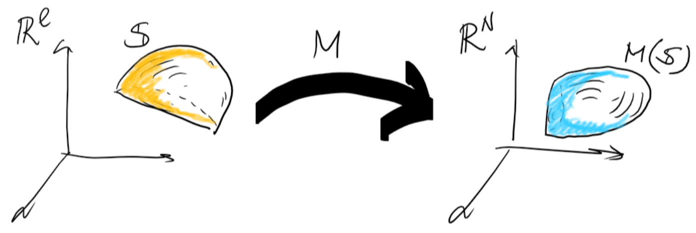

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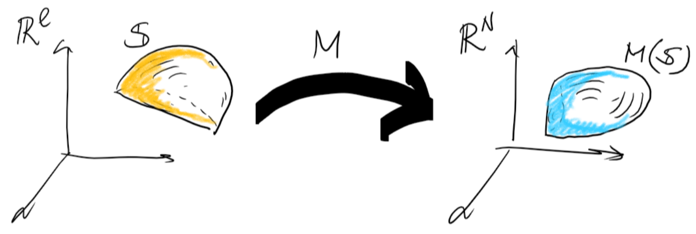

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- **Theorem:** the range $M(\mathbb{S})$ identifies M up to gauge symmetries

Quiz

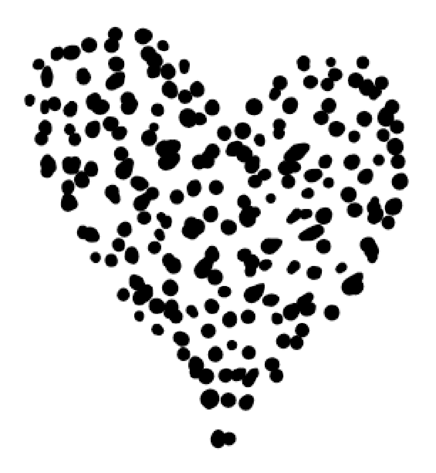
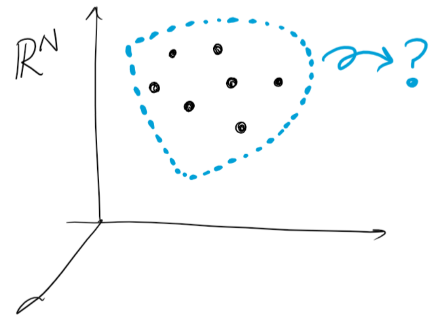


Figure 1: What do you see?

Inferring a Range from the Dataset



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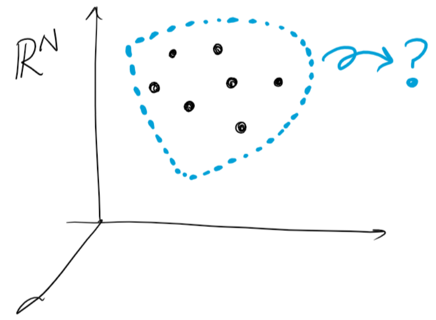

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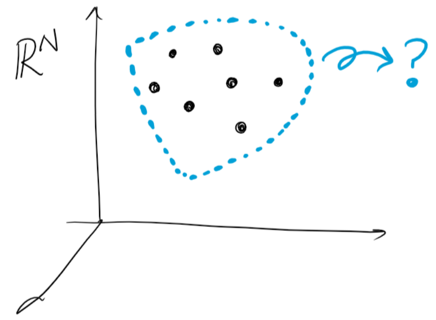

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- **Data-Driven Inference (DDI) Rule:** in the face of data $D = \{\boldsymbol{p}_x \in \mathbb{R}^N\}$, infer the range which:
  1. contains the convex hull of D and
  2. is of minimum euclidean volume

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- **Problem 1:** in order to apply **DDI**, one first needs to know the shape of $\mathbb{S}$
- **Problem 2:** empirical data are not probability distributions but finite-statistics frequencies

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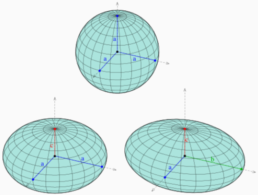
- Question: are there less demanding OC sets?
- **Theorem:** $\mathcal{S} \subseteq \mathbb{S}$ is observationally complete *for any* measurement whenever **DDI**$[\mathcal{S}] = \mathbb{S}$

The special case of spherical theories

When $\mathbb{S}$ Is a Hypersphere...



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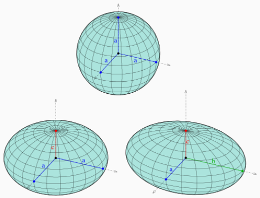

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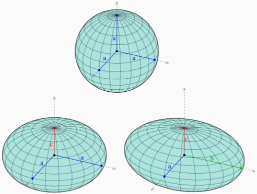

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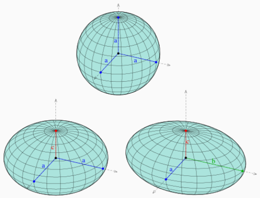

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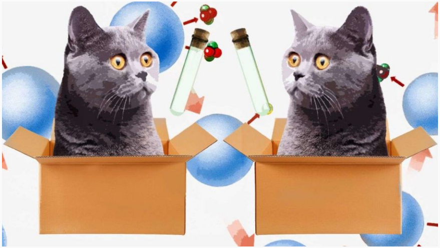
- in fact, hyperspherical theories are *exactly those* that allow a unique inference *for any* dataset
- **DDI** may still return an ellipsoid which is not the range of a valid measurement: in this case a failure is announced

The Case of Qubits



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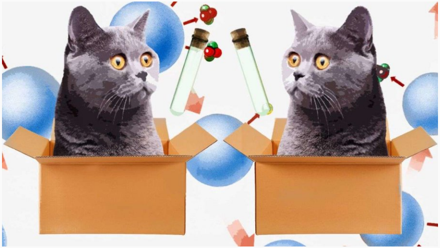

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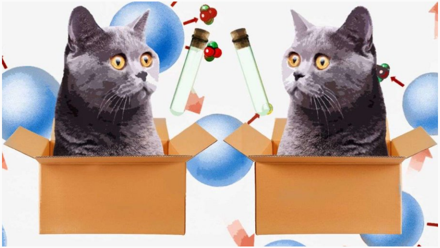

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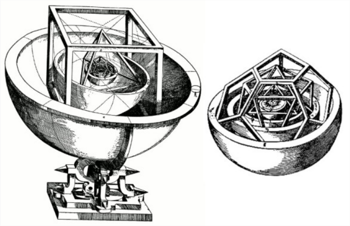
- hence, **DDI** is able to return a qubit measurement up to unitaries or antiunitaries
- moreover, a representative measurement can be explicitly constructed for any range (closed formula)

Observationally Complete Sets for Qubits



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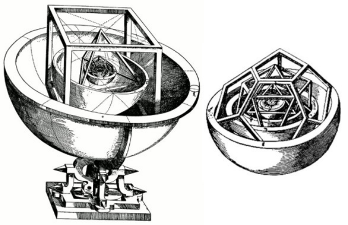

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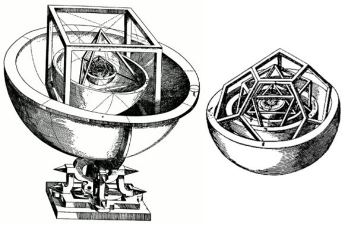

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- hence, SIC ensembles are OC

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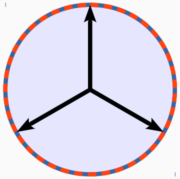
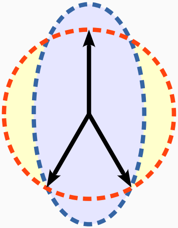


Figure 2: A regular simplex is OC.



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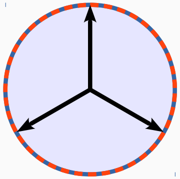

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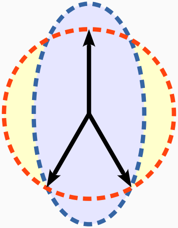

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In particular:

- a pure SIC ensemble is also OC

Example: Observational VS Informational Completeness

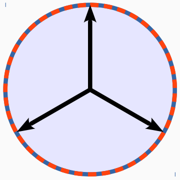

Figure 2: A regular simplex is OC.

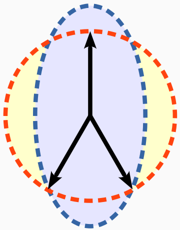

Figure 3: An irregular simplex is not OC.

In particular:

- a pure SIC ensemble is also OC
- a depolarized SIC ensemble still is IC and "symmetric" but is not OC anymore

Example: DDI in Action

- suppose the dataset comprises three probability distributions in $\mathbb{R}^4$, that is $D = \{\boldsymbol{p}_1 = (\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}), \boldsymbol{p}_2 = (\frac{1}{8}, \frac{3}{8}, \frac{2+\sqrt{3}}{8}, \frac{2-\sqrt{3}}{8}), \boldsymbol{p}_3 = (\frac{1}{8}, \frac{3}{8}, \frac{2-\sqrt{3}}{8}, \frac{2+\sqrt{3}}{8})\}$

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- **DDI:** the four effects are coplanar and arranged in a square

Conclusions

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References:

- initial idea (case of qubit channels): F.B. and M. Dall'Arno. `arXiv:1805.01159`
- experiment: I. Agresti, D. Poderini, G. Carvacho, L. Serra, R. Chaves, F.B., M. Dall'Arno, F. Sciarrino. `arXiv:1806.00380`
- this talk: M. Dall'Arno, F.B., A. Bisio, A. Tosini. `arXiv:1812.08470`
- M. Dall'Arno, A. Ho, F.B., V. Scarani. `arXiv:1905.04895`

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thank you