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Self-testing quantum systems of arbitrary local dimension

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A. Salavrakos, R.A., J. Tura, P. Wittek, A. Acín, S. Pironio, PRL **119**, 040402 (2017)

J. Kaniewski, I. Šupić, J. Tura, F. Baccari, A. Salavrakos, R.A., arXiv:1807.06027

D. Saha, S. Sarkar, J. Kaniewski, R.A., under construction

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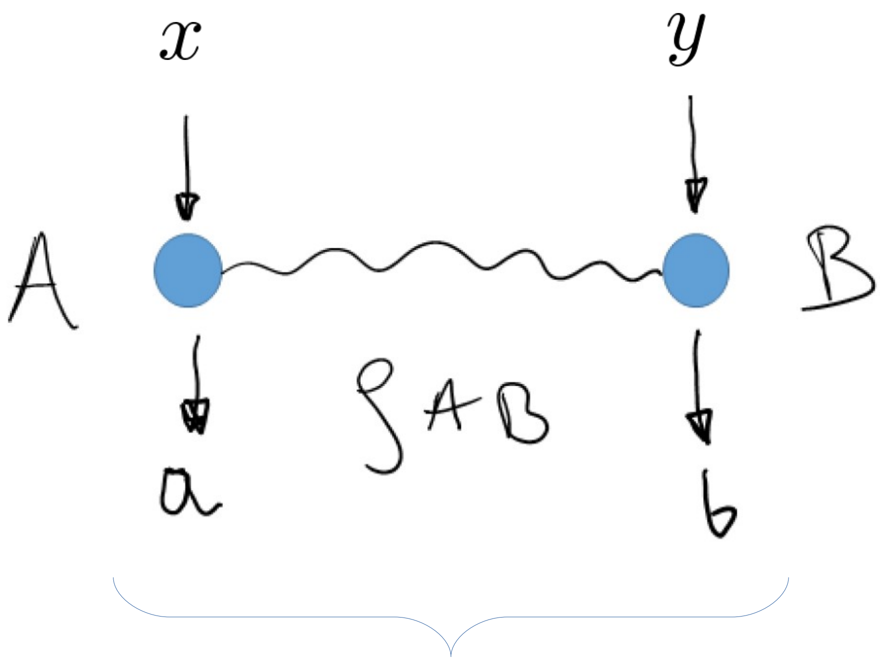
Preliminaries

(2,m,d) scenario

- **Bell scenario:** two parties performing measurements on their local systems



$$\{p(a,b|x,y)\}$$

$$p(a,b|x,y) = \mathrm{Tr}[\rho_{AB}(M_x^a \otimes N_y^b)]$$

measurement choices

$$x, y = 0, \ldots, m-1$$

outcomes

$$a, b = 0, \ldots, d-1$$

measurement $M = \{M_a\}$

$$M_a \geq 0, \;\; \sum_a M_a = \mathbb{1}$$

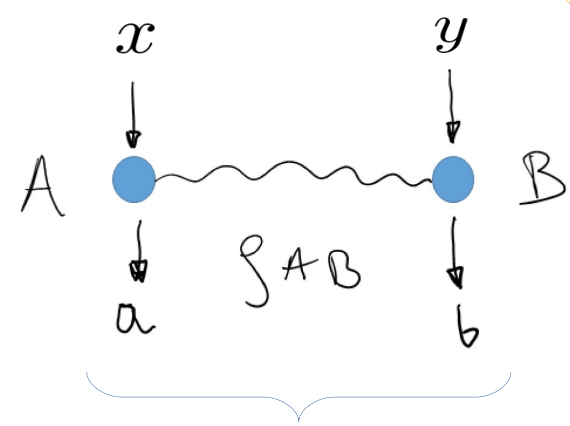
(POVM/generalized measurement)

$$M_a M_{a'} = \delta_{a,a'} M_a$$

(Projective measurement)

Preliminaries

- **Correlations:** described by a set of probability distributions



$$p(a,b|x,y)$$

$$\{p(a,b|x,y)\}$$

$$x,y = 0,\dots,m-1$$

$$a,b = 0,\dots,d-1$$

- Alternatively by a set of **generalized expectation values**

$$\langle A_x^k B_y^l \rangle = \sum_{a,b=0}^{d-1} \omega^{ak+bl} p(a,b|x,y)$$

2D Fourier transform of $\{p(a,b|x,y)\}$

$$\langle A_x^k B_y^l \rangle = \mathrm{Tr}[\rho_{AB}(A_x^{(k)} \otimes B_y^{(l)})]$$

$$A_x^{(k)} = \sum_{a=0}^{d-1} \omega^{ak} M_x^a$$

A_x, B_y: unitary operators with eigenvalues $1,\omega,\dots,\omega^{d-1}$

$$\omega = \exp(2\pi \mathbb{i}/d)$$

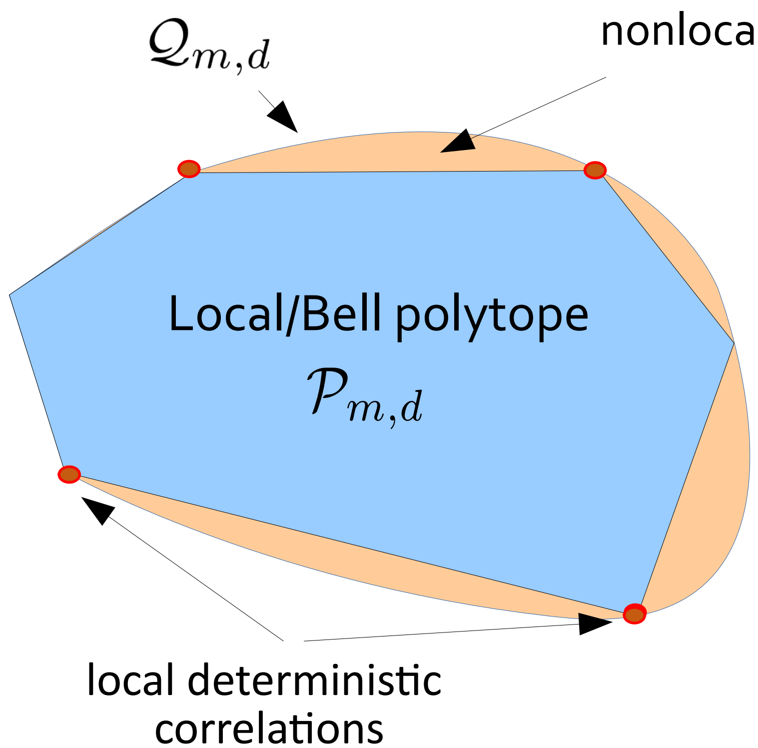
[J. S. Bell, Physics **1**, 195 (1964)]

- Local/classical correlations

$$p(a,b|x,y) = \sum_{\lambda} p(\lambda) p_{\det}(a|x,\lambda) p_{\det}(b|y,\lambda)$$

$$\forall_{a,b,x,y,\lambda} \quad \left.\begin{matrix} p_{\det}(a|x,\lambda) \\ p_{\det}(b|y,\lambda) \end{matrix}\right\} \in \{0,1\}$$

λ – hidden variable



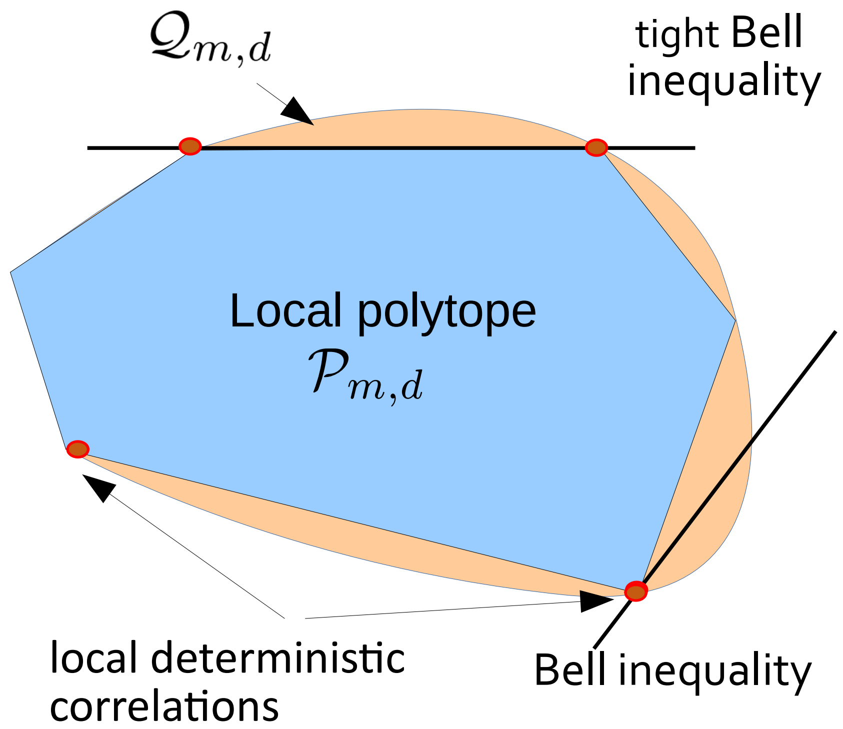
$$\vec{p} = \{p(ab|xy)\} \in \mathbb{R}^D$$

$$D = (dm)^2$$

- Otherwise they are called **nonlocal** ⟶ **nonlocality**

$$\mathcal{P}_{m,d} \subsetneq \mathcal{Q}_{m,d}$$

- **Bell inequalities**: Hyperplanes constraining the local set



$$I := \sum_{a,b,x,y} T_{x,y}^{a,b}\, p(a,b|x,y) \leq \beta_C$$

$$\beta_C = \max_{\mathcal{P}_{m,d}} I \qquad \text{(classical bound)}$$

$$\beta_Q = \sup_{\mathcal{Q}_{m,d}} I \qquad \text{(quantum bound)}$$

- Finite number of BI's (facets) enough to fully characterize the local polytope → tight Bell inequalities (convex hull problem)
- The number of vertices $= d^{2m}$ → difficult task

Examples

Clauser, Horne, Shimony, Holt (1969);
Collins *et al.* (CGLMP) (2002);
Barrett, Kent, Pironio (BKP) (2006);

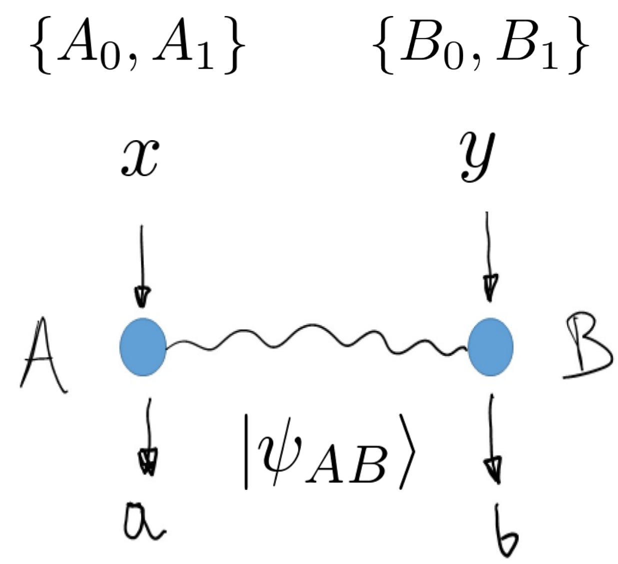
Preliminaries

CHSH Bell inequality

[Clauser, Horne, Shimony, Holt (1969)]

- **Example**: the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2$$



$a, b = \pm 1$

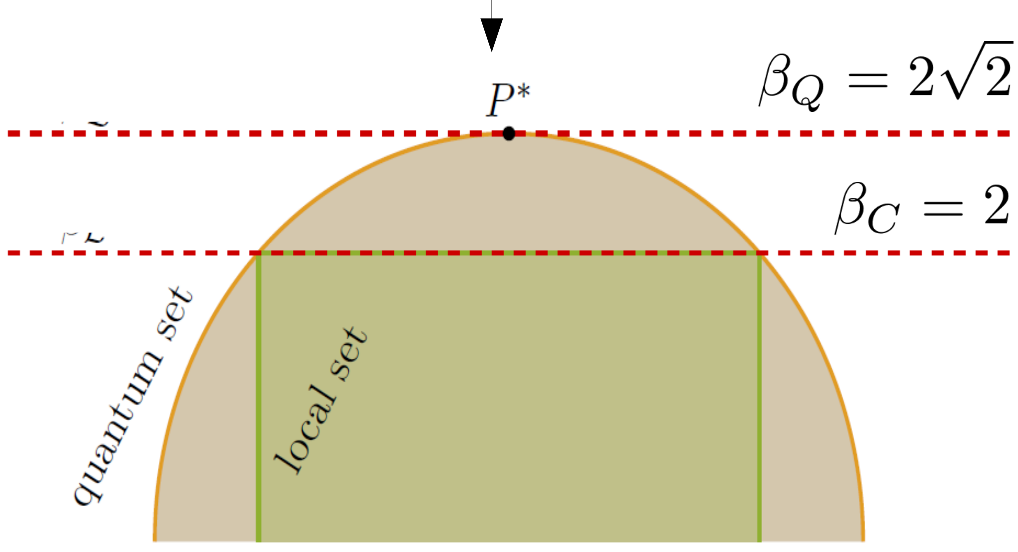
Maximal quantum violation

$$|\psi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$A_0 = \sigma_x \quad B_0 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

$$A_1 = \sigma_z \quad B_1 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_z)$$

mutually unbiased bases (MUB)



Non-locality

- Non-locality is a resource for device-independent applications

- Quantum key distribution
  [Ekert, PRL (1991); A. Acín *et al.*, PRL (2007)]

- Randomness certification/amplification
  [Pironio *et al.*, Nature (2010); Colbeck, Renner, Nat. Phys. (2012)]

- Device-independent entanglement certification
  [J.-D. Bancal *et al.*, PRL (2011)]

- Self-testing
  [Mayers, Yao, QIC (2004)]

[Mayers, Yao, 2004]

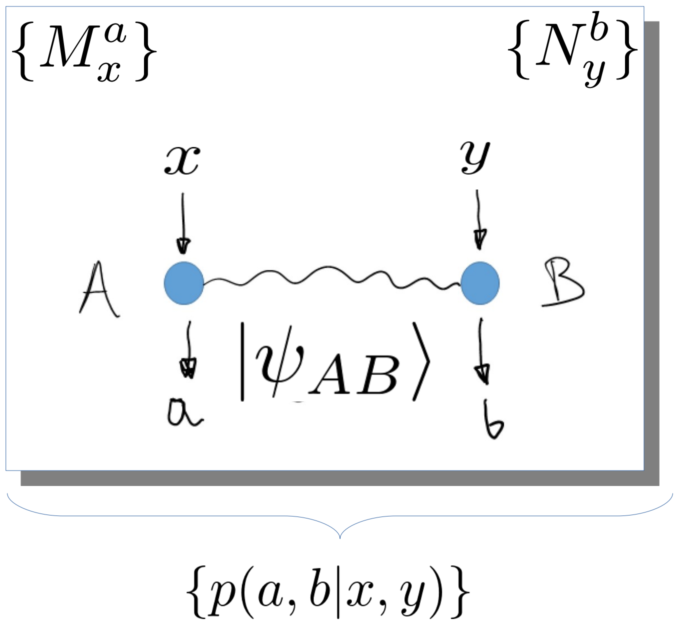
The idea of device-independent certification

- Given $\{p(a,b|x,y)\}$
- or violation of some Bell inequality

$$\sum_{a,b,x,y} T_{x,y}^{a,b}\, p(a,b|x,y) = \beta > \beta_C$$

- **deduce properties of** $|\psi_{AB}\rangle$ **and** $\{M_x^a\},\ \{N_y^b\}$

Quantum device



Self-testing

[Mayers, Yao, 2004]

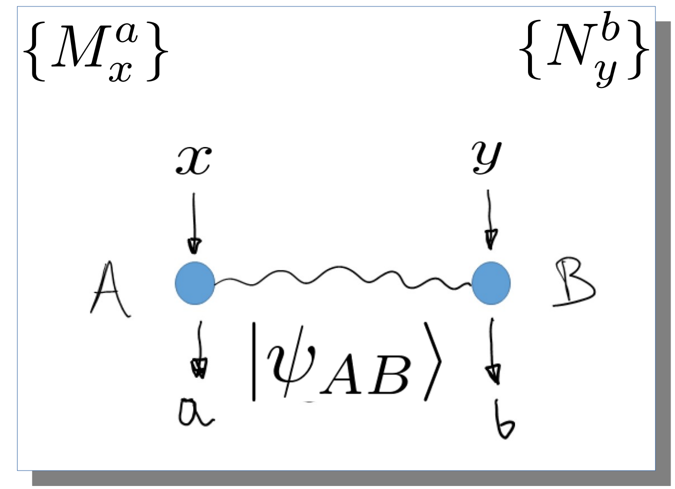
- **The idea of device-independent certification**

  - Given [illegible]
  - or violation of some Bell inequality

$$\sum_{a,b,x,y} T_{x,y}^{a,b}\, p(a,b|x,y) = \beta > \beta_C$$

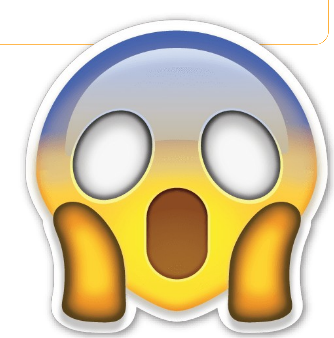
  - **deduce properties of** $|\psi_{AB}\rangle$ **and** $\{M_x^a\},\ \{N_y^b\}$

Quantum device



- **Seems like a hopeless task!**

maybe not …
often one can deduce **everything**!



Self-testing

[Tsirelson 93; S. Popescu, D. Rohrlich, 92]

▶ **Example**: self-testing statement for CHSH

Assume that $I_{\text{CHSH}} = 2\sqrt{2}$ for $\underbrace{|\psi_{AB}\rangle, A_x, B_y}_{\text{unknown}}$

⇓

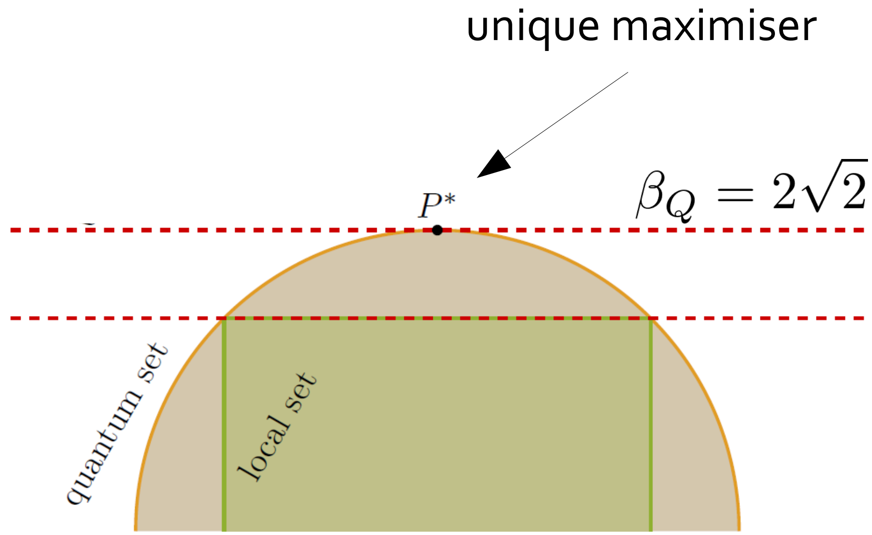
$$U_A\, A_0\, U_A^\dagger = \sigma_x \otimes \mathbb{1} \qquad U_B\, B_0\, U_B^\dagger = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) \otimes \mathbb{1} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_A\, A_1\, U_A^\dagger = \sigma_z \otimes \mathbb{1} \qquad U_B\, B_1\, U_B^\dagger = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_z) \otimes \mathbb{1} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$U_A \otimes U_B |\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |\text{aux}\rangle$$

- The CHSH Bell inequality self-tests the maximally entangled state and the above observables

unique quantum realisation
(up to local unitary operations)



Self-testing

- All **two-qubit** pure entangled states

  Tsirelson 93, S. Popescu, D. Rohrlich, 1992, Meyers, Yao, 2004;
  M. McKague *et al.*, 2012; Yang, Navascués 2013;
  J. Kaniewski, 2017

- Maximally entangled states of two qudits

  Yang, Navascués, 2014

- All bipartite pure entangled states

  A. Coladangelo *et al.*, 2017

Self-testing from projections
+
the CHSH Bell inequality

$$|\psi_3^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

$$|00\rangle + |11\rangle$$

$$|11\rangle + |22\rangle$$

$$|00\rangle + |22\rangle$$

self-testing from the corresponding CHSH inequality

Problems

- **Problem:** Self-testing from genuinely d-outcome Bell inequalities

Quantum states — Measurements

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle_A\otimes|i\rangle_B \in \mathbb{C}^d\otimes\mathbb{C}^d$$

- perfect correlations in any local basis (QKD)
- maximizer of entanglement measures
- $\rho_A = \rho_B = \mathbb{1}/d$ (randomness certification)

Problems

Problem: Self-testing from genuinely *d*-outcome Bell inequalities

Quantum states

Measurements

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B \in \mathbb{C}^d \otimes \mathbb{C}^d$$

- perfect correlations in any local basis (QKD)
- maximizer of entanglement measures
- $\rho_A = \rho_B = \mathbb{1}/d$ (randomness certification)

Another problem: no general class of Bell inequalities for maxent quantum states

- CHSH (1971) – (2,2,2) scenario
- CGLMP (2001), BKP (2006)
- Buhrman, Massar (2005)
- Son *et al.* (2006) – (2,2,d) scenario
- Ji *et al.* (2008) , Liang *et al.* (2009)

- the maximal quantum violation β_Q unknown
- violated maximally by nonmaximally entagled states

Bell inequalities tailored to the maxent states

- **Inequality I:** Modification of the famous CGLMP Bell expression

[Collins *et al.*, PRL (2002); Barrett *et al.*, PRL (2006)]

- *(2,m,d)* scenario
- Self-testing statement for any *d*

A. Salavrakos, R.A., J. Tura,
P. Wittek, A. Acín, S. Pironio,
PRL **119**, 040402 (2017)

D. Saha, S. Sarkar, J. Kaniewski, R.A.
under construction

- **Inequality II:** Modification of the CHSH-*d* inequality

[Buhrman, Massar, (2005)]

- *(2,d,d)* scenario with prime *d*
- Self-testing statement for *d=3*

J. Kaniewski, I. Šupić, J. Tura,
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Bell inequalities tailored to the maxent states

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  J. Kaniewski, I. Šupić, J. Tura, F. Baccari, A. Salavrakos, R.A., arXiv:1807.06027

Results

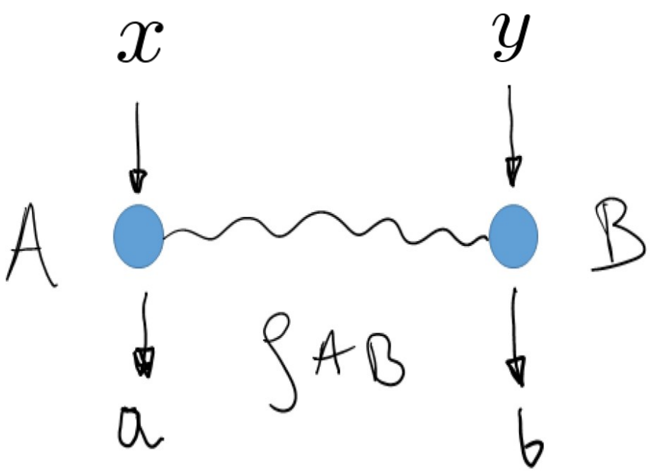
[arXiv:1807.03332]

▸ **Result I**: Bell inequalities maximally violated by the maxent state and MUBs

▸ The original CHSH inequality [(2,2,2) scenario]

$$I_2 := \sum_{x,y=0}^{1} (-1)^{xy} \langle A_x B_y \rangle \leq 2$$

nonlocal game $\Leftrightarrow$ $\begin{cases} x + y + a \cdot b = 0 \pmod 2 \\ a, b, x, y \in \{0, 1\} \end{cases}$



▸ A generalisation to *d* outcomes – CHSH-*d* inequality [(2,*d*,*d*) scenario]

$$I_d := \sum_{n=1}^{d-1} \sum_{x,y=0}^{d-1} \omega^{nxy} \langle A_x^n B_y^n \rangle \leq \beta_C \quad \Leftrightarrow \quad \begin{cases} x + y + a \cdot b = 0 \pmod d \\ a, b, x, y \in \{0, 1, \ldots, d-1\} \end{cases}$$

A_x, B_y – unitary observables with eigenvalues $1, \omega, \ldots, \omega^{d-1}$ $[\omega = \exp\left(\frac{2\pi \mathrm{i}}{d}\right)]$

Buhrman, Massar, (2005);
Ji et al., (2008); Bavarian, Shor (2013),
Liang et al. (2009)]

Results

[arXiv:1807.03332]

- Modifying the CHSH-*d* inequality [prime *d*]

$$\widetilde{I}_d := \sum_{n=1}^{d-1} \underbrace{\lambda_{n,d}} \sum_{x,y=0}^{d-1} \omega^{nxy} \langle A_x^n B_y^n \rangle \leq \beta_C$$

phases chosen so that $\frac{\lambda_{n,d}}{\sqrt{d}} A_i^n \otimes C_i^{(n)} |\psi_d\rangle = |\psi_d\rangle$

$$|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i,i\rangle$$

$$C_i^{(n)} = \sum_j \omega^{nij} B_j^n$$

$$B_k = \omega^{k(k+1)} X Z^k$$

$$A_k = [C_i^{(0)}]^*$$

mutually unbiased bases in $\mathbb{C}^d$

[Ji et al. (2008)]

- Easier to characterise: direct computation of the max. quantum value

$$\beta_Q = \max_{\mathcal{Q}} \tilde{I}_d = d\sqrt{d}\,(d-1)$$

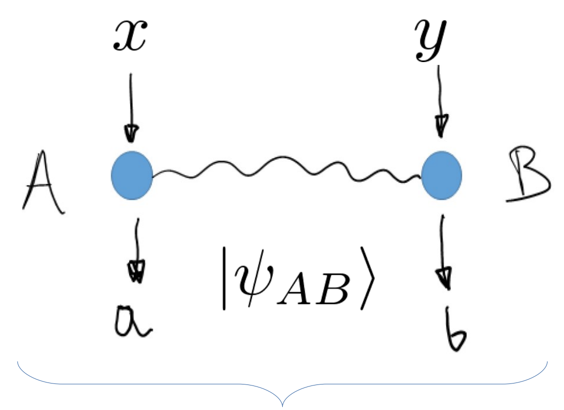
$$\beta_Q \mathbb{1} - \mathcal{B} \sim \sum_{n,i} L_i^{(n)\dagger} L_i^{(n)} \qquad L_i^{(n)} = \mathbb{1} \otimes \mathbb{1} - A_i^n \otimes C_i^{(n)}$$

Results

[arXiv:1807.03332]

- For *d*=3 our inequality self-tests the maximally entangled state and MUBs!

$\{A_0, A_1, A_2\}$ $\{B_0, B_1, B_2\}$



- Let $\underbrace{|\psi_{AB}\rangle, A_x, B_y}_{\text{unknown}}$ violate our inequality maximally

$a, b \in \{1, \omega, \omega^2\}$

- $\mathbb{C}^D = \mathbb{C}^3 \otimes \mathbb{C}^{d'}$
- $\exists U_B \quad \text{s.t.} \quad U_B\, B_0\, U_B^\dagger = X \otimes \mathbb{1}_{d'}$

$$U_B\, B_1\, U_B^\dagger = X^2 Z \otimes \mathbb{1}_{d'}$$

$$U_B\, B_2\, U_B^\dagger = Z^2 \otimes \mathbb{1}_{d'}$$

(+ their transpositions)

- $U_A \otimes U_B |\psi_{AB}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \otimes |\text{ancilla}\rangle$

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- For *d*>3 the problem complicates significantly

$$\omega^q B_0 B_1 = B_1 B_0 \quad (q = 1, \ldots, d-1)$$

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

Bell inequalities tailored to the maxent states

- **Inequality I:** Modification of the famous CGLMP Bell expression

  [Collins *et al.*, PRL (2002); Barrett *et al.*, PRL (2006)]

  - *(2,m,d)* scenario

    A. Salavrakos, R.A., J. Tura, P. Wittek, A. Acín, S. Pironio, PRL **119**, 040402 (2017)

  - Self-testing statement for any *d*

    D. Saha, S. Sarkar, J. Kaniewski, R.A. under construction

Constructing Bell inequalities

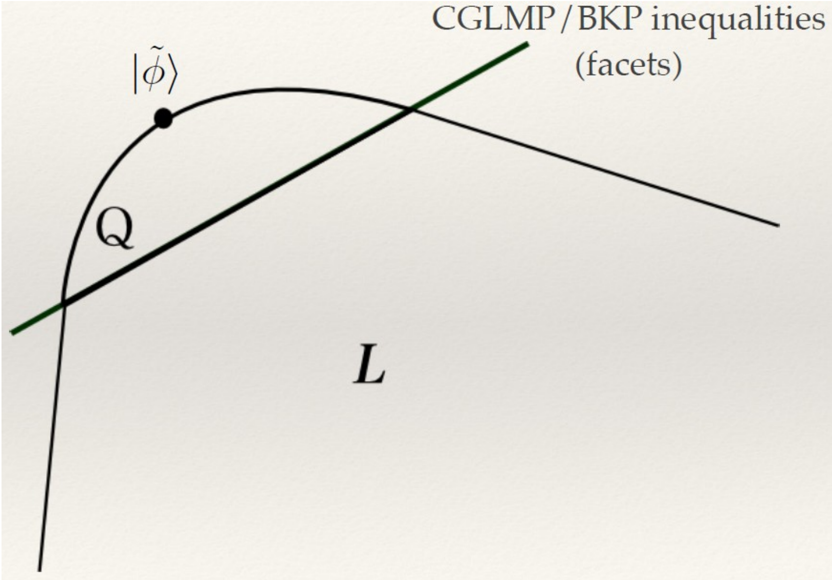
Phys. Rev. Lett. 119, 040402 (2017)

- Consider the Barrett-Kent-Pironio (BKP) Bell expression

[Collins *et al.*, PRL (2002); Barrett *et al.*, PRL (2006)]

$$I_{d,m} := \sum_{k=0}^{\lfloor d/2 \rfloor - 1} (\mathbb{P}_k - \mathbb{Q}_k)$$

$p(a,b|x,y)$



- facet Bell inequalities in *(2,2,d)* scenario

[Masanes, QIC (2002)]

- not maximally violated by $|\psi_d^+\rangle$
- e.g., for *d=3*

$$|\widetilde{\phi}\rangle \sim |00\rangle + \gamma|11\rangle + |22\rangle \qquad \gamma = \tfrac{1}{2}(\sqrt{11} - \sqrt{3})$$

[Acin, Durt, Gisin, QIC (2002); Yang *et al.* (2014)]

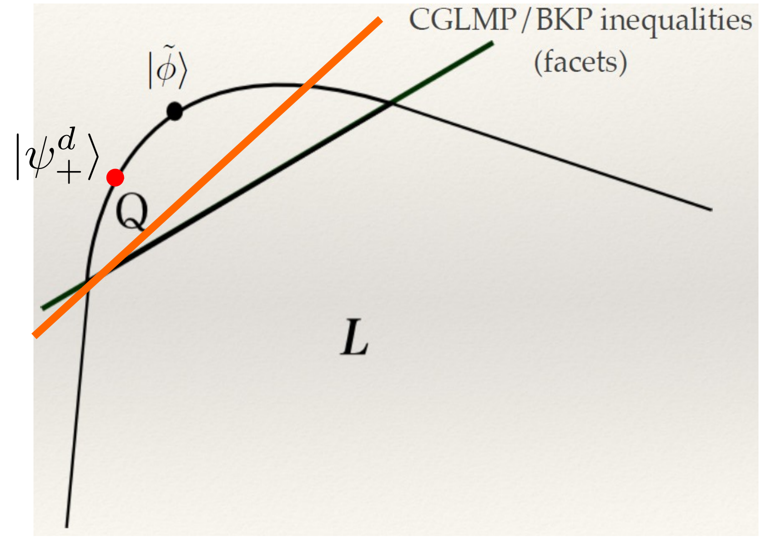
Constructing Bell inequalities

Phys. Rev. Lett. 119, 040402 (2017)

- Modify by adding parameters (tilting the inequality)

$$I_{d,m} := \sum_{k=0}^{\lfloor d/2 \rfloor - 1} (\alpha_k \mathbb{P}_k - \beta_k \mathbb{Q}_k)$$

$$\alpha_k, \beta_k \in \mathbb{R}$$



- "Quantum approach" (CHSH inspired)

$$\widetilde{I}_{d,m} := \sum_{i=1}^{m} \sum_{l=1}^{d-1} \left(a_l \langle A_i^l B_i^{d-l} \rangle + a_l^* \langle A_i^l B_{i-1}^{d-l} \rangle \right) = \sum_{i=1}^{m} \sum_{l=1}^{d-1} \langle A_i^l C_i^{(l)} \rangle \qquad a_l \equiv a_l(\alpha_k, \beta_k)$$

$$C_i^{(l)} = a_l B_i^{d-l} + a_l^* B_{i-1}^{d-l}$$

$$A_i^l \otimes C_i^{(l)} |\psi_+^d\rangle = |\psi_+^d\rangle \quad \Rightarrow \quad \alpha_k, \beta_k$$ almost uniquely

A_i, B_i optimal CGLMP/BKP measurements

[Collins *et al.* (CGLMP) (2002);
Barrett, Kent, Pironio (BKP) (2006)]

Full characterization of our Bell inequalities

Phys. Rev. Lett. 119, 040402 (2017)

- Analytical proof of the maximal quantum value

$$\beta_Q = \max_{\mathcal{Q}_{m,d}} \tilde{I}_{m,d} = m(d-1)$$

- quantum realization $|\psi_d^+\rangle$ and optimal CGLMP measurements

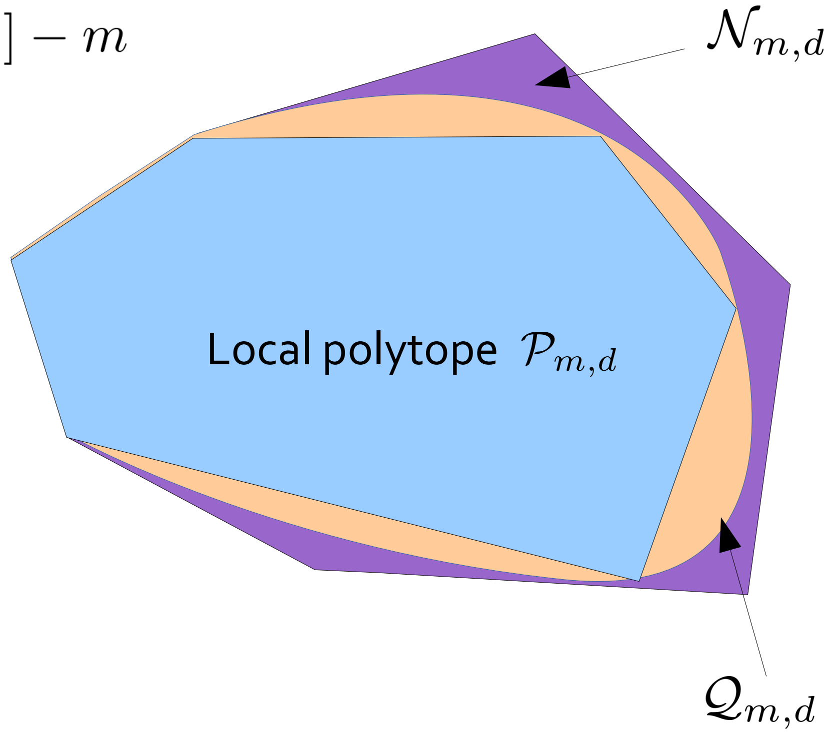
- Analytical computation of the classical bound

$$\begin{aligned}\beta_C &= \max_{\mathcal{P}_{m,d}} \tilde{I}_{m,d} \\ &= \tfrac{1}{2}\tan\left(\tfrac{\pi}{2m}\right)\left[(2m-1)g(0) - g(1-1/m)\right] - m\end{aligned}$$

- The maximal nonsignaling value

$$\beta_{NS} = \max_{\mathcal{N}_{m,d}} \tilde{I}_{m,d} = m\tan\left(\tfrac{\pi}{2m}\right) g(0) - m$$

$$g(x) = \cot\left[\tfrac{\pi}{d}\left(x + \tfrac{1}{2m}\right)\right]$$



- Asymptotic properties of β_Q/β_C and β_{NS}/β_Q

Self-testing with SATWAP inequality

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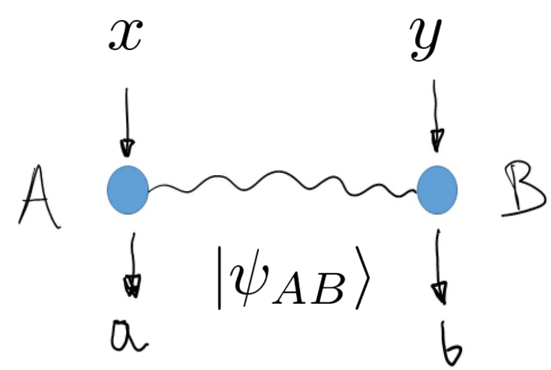
under construction

- $|\psi_{AB}\rangle \in \mathbb{C}^D \otimes \mathbb{C}^D$ and $A_x,\ B_y$ maximize $\widetilde{I}_{d,2}$

$\{A_0, A_1\} \qquad \{B_0, B_1\}$



↓

- $\mathbb{C}^D = \mathbb{C}^d \otimes \mathbb{C}^{d'}$
- $\exists U_B \quad \text{s.t.} \quad U_B\, B_0\, U_B^\dagger = Z \otimes \mathbb{1}_{d'}$

  $U_B\, B_1\, U_B^\dagger = T \otimes \mathbb{1}_{d'}$
- $\exists U_A \quad \text{s.t.} \quad U_A\, A_0\, U_A^\dagger = (aZ + bT) \otimes \mathbb{1}_{d'}$

  $U_A\, A_1\, U_A^\dagger = (a'Z^\dagger + b'T) \otimes \mathbb{1}_{d'}$
- $U_A \otimes U_B\, |\psi_{AB}\rangle = |\psi_d^+\rangle \otimes |\text{aux}\rangle$

$a, b \in \{1, \omega, \ldots, \omega^{d-1}\}$

$Z,\ T$ – $d \times d$ unitary matrices

$$Z = \sum_{i=0}^{d-1} \omega^i |i\rangle\langle i|$$

$|\text{aux}\rangle \in \mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$

(+ global transposition)

Self-testing with SATWAP inequality

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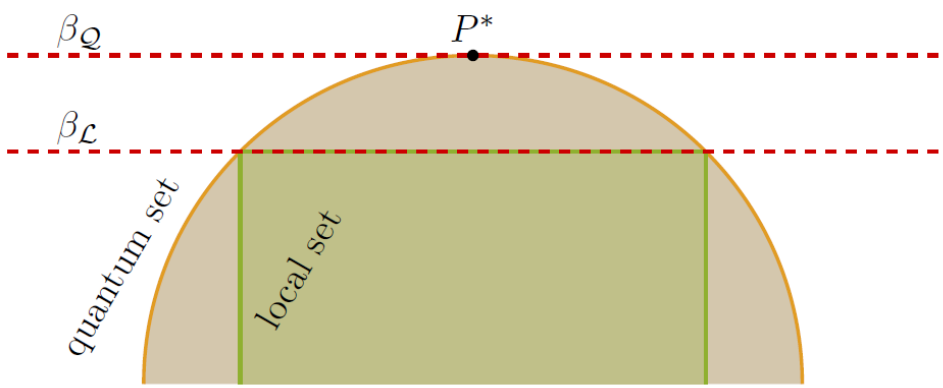
under construction

- **Corollary 1:** Our Bell expression has a unique maximiser



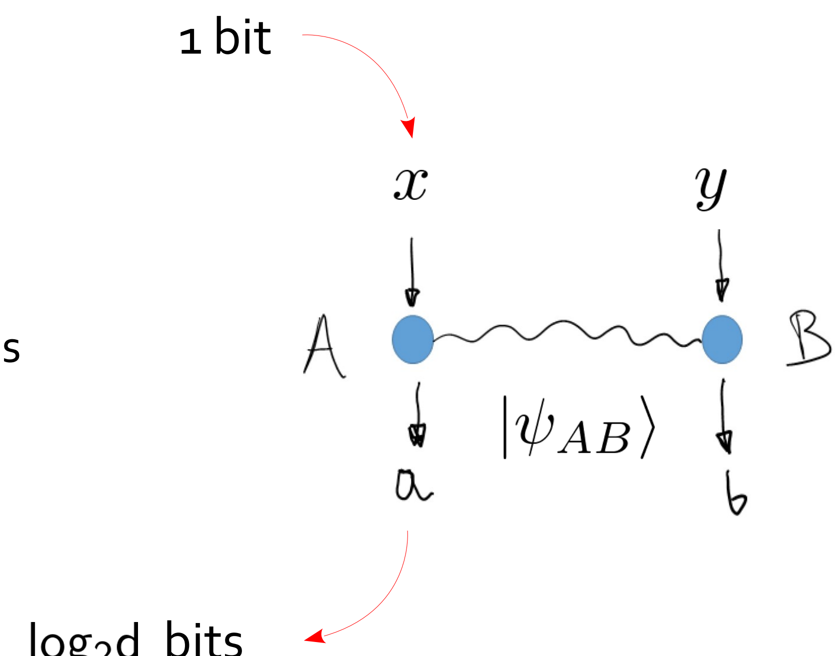
- **Corollary 2:** Maximal violation of our inequality certifies $\log_2 d$ bits of local randomness

for $P^* = \{p(a,b|x,y)\}$

$$\forall_{a,x}\ p_A(a|x) = \frac{1}{d}$$

$$\forall_{b,y}\ p_B(b|y) = \frac{1}{d}$$

⇒ unbouded randomness expansion



[Skrzypczyk, Cavalcanti, PRL 2018]

Self-testing with SATWAP inequality

- Sum of squares decomposition (case *m*=2)

$$\beta_Q \mathbb{1} - \mathcal{B}_{d,2} = \frac{1}{2}\sum_{k=1}^{d-1}\left[P_{1,k}^{\dagger}P_{1,k} + P_{2,k}^{\dagger}P_{2,k}\right]$$

$$\beta_Q = 2(d-1)$$

$$P_{i,k} = \mathbb{1} - A_i^k \otimes C_i^{(k)}$$

$$C_0^{(k)} = a_k B_0^{d-k} + a_k^* \omega^k B_1^{d-k}$$

$$C_1^{(k)} = a_k B_1^{d-k} + a_k^* B_0^{d-k}$$

- Sketch of the proof

$\tilde{I}_{d,2} = 2(d-1)$ for $|\psi_{AB}\rangle$ and $A_x,\ B_y$

⇕

$$[\beta_Q \mathbb{1} - \mathcal{B}_{d,2}]|\psi_{AB}\rangle = 0$$

⇕ sum of squares

$$P_{i,k}|\psi_{AB}\rangle = 0 \qquad i = 1,2;\ k = 1,\ldots,d-1$$

Self-testing with SATWAP inequality

- Sketch of the proof – *cd*

$$A_i^k \otimes C_i^{(k)} |\psi_{AB}\rangle = |\psi_{AB}\rangle \qquad i = 1, 2;\; k = 1, \ldots, d-1$$

⇕

$$C_i^{(k)} = [C_i^{(1)}]^k, \qquad [C_i^{(k)}]^\dagger [C_i^{(k)}] = \mathbb{1}_D$$

on the support of

$$\rho_B = \mathrm{Tr}_A |\psi_{AB}\rangle\langle\psi_{AB}|$$

⇓

for some *k's* $\quad \mathrm{Tr}(B_i^k) = 0$

eigenvalues of B_i have the same multiplicities for d prime and $d = 2^n$

⇓

$\exists U_B$ s.t. $\quad U_B\, B_0\, U_B^\dagger = Z \otimes \mathbb{1}_{d'}$ ⇒ $U_B\, B_1\, U_B^\dagger = T \otimes \mathbb{1}_{d'}$

the same for Alice

Conclusion/Outlook

- Two classes of Bell inequalities maximally violated by the maxent states of local dimension higher than 2
- Self-testing statement for d>2 with the minimal amount of measurements
- Unbounded randomness expansion from quantum correlations

- Make our self-testing statements robust

$$\widetilde{I}_{d,2} = \beta_Q - \epsilon \quad \overset{?}{\Longrightarrow} \quad \| |\psi_{AB}\rangle - |\psi_d^+\rangle \otimes |\mathrm{aux}\rangle \| \leq f(\epsilon) \xrightarrow{\epsilon \to 0} 0$$

- Explore whether our results can be generalized

partially entangled states | various measurements

- Generalization to the multiparty scenario (*work in progress*)

F. Baccari, R. A., I. Šupić, J. Tura, A. Acin, arXiv:1812.10428

GHZ states | graph states, AME states | All entangled multipartite states