which is related to the gamma function by

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \qquad (6.1.9)$$

hence

```
#include <math.h>

float beta(float z, float w)
Returns the value of the beta function B(z, w).
{
    float gammln(float xx);

    return exp(gammln(z)+gammln(w)-gammln(z+w));
}
```

CITED REFERENCES AND FURTHER READING:

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, vol. 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapter 6.

Lanczos, C. 1964, *SIAM Journal on Numerical Analysis*, ser. B, vol. 1, pp. 86–96. [1]

6.2 Incomplete Gamma Function, Error Function, Chi-Square Probability Function, Cumulative Poisson Function

The incomplete gamma function is defined by

$$P(a,x) \equiv \frac{\gamma(a,x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \qquad (a > 0) \qquad (6.2.1)$$

It has the limiting values

$$P(a,0) = 0 \qquad \text{and} \qquad P(a,\infty) = 1 \qquad (6.2.2)$$

The incomplete gamma function $P(a,x)$ is monotonic and (for a greater than one or so) rises from “near-zero” to “near-unity” in a range of x centered on about $a-1$, and of width about $\sqrt{a}$ (see Figure 6.2.1).

The complement of $P(a,x)$ is also confusingly called an incomplete gamma function,

$$Q(a,x) \equiv 1 - P(a,x) \equiv \frac{\Gamma(a,x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt \qquad (a > 0) \qquad (6.2.3)$$

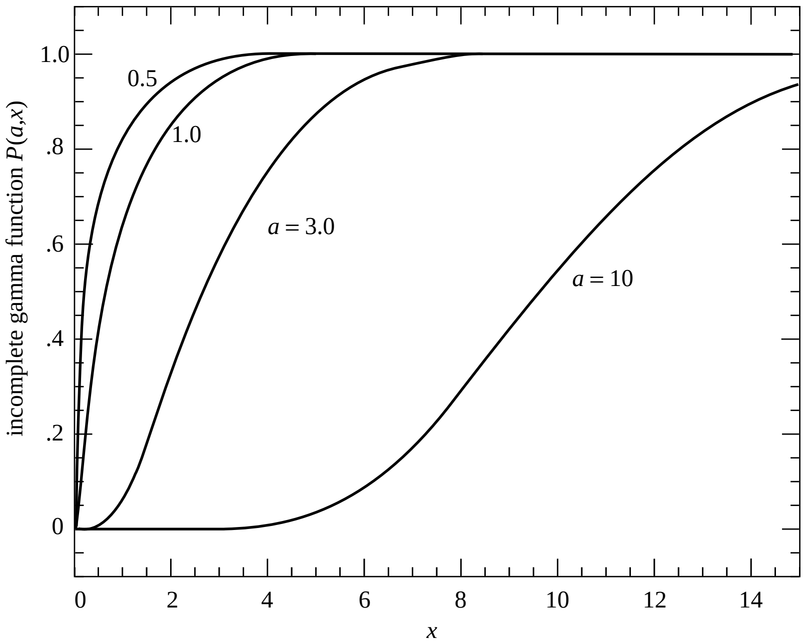

Figure 6.2.1. The incomplete gamma function $P(a, x)$ for four values of a.

It has the limiting values

$$Q(a, 0) = 1 \qquad \text{and} \qquad Q(a, \infty) = 0 \tag{6.2.4}$$

The notations $P(a, x), \gamma(a, x)$, and $\Gamma(a, x)$ are standard; the notation $Q(a, x)$ is specific to this book.

There is a series development for $\gamma(a, x)$ as follows:

$$\gamma(a, x) = e^{-x} x^a \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a + 1 + n)} x^n \tag{6.2.5}$$

One does not actually need to compute a new $\Gamma(a + 1 + n)$ for each n; one rather uses equation (6.1.3) and the previous coefficient.

A continued fraction development for $\Gamma(a, x)$ is

$$\Gamma(a, x) = e^{-x} x^a \left(\frac{1}{x+} \; \frac{1-a}{1+} \; \frac{1}{x+} \; \frac{2-a}{1+} \; \frac{2}{x+} \cdots \right) \qquad (x > 0) \tag{6.2.6}$$

It is computationally better to use the even part of (6.2.6), which converges twice as fast (see §5.2):

$$\Gamma(a, x) = e^{-x} x^a \left(\frac{1}{x + 1 - a -} \; \frac{1 \cdot (1 - a)}{x + 3 - a -} \; \frac{2 \cdot (2 - a)}{x + 5 - a -} \; \cdots \right) \qquad (x > 0) \tag{6.2.7}$$

It turns out that (6.2.5) converges rapidly for x less than about $a + 1$, while (6.2.6) or (6.2.7) converges rapidly for x greater than about $a+1$. In these respective regimes each requires at most a few times $\sqrt{a}$ terms to converge, and this many only near $x = a$, where the incomplete gamma functions are varying most rapidly. Thus (6.2.5) and (6.2.7) together allow evaluation of the function for all positive a and x. An extra dividend is that we never need compute a function value near zero by subtracting two nearly equal numbers. The higher-level functions that return $P(a, x)$ and $Q(a, x)$ are

```
float gammp(float a, float x)
Returns the incomplete gamma function P(a, x).
{
    void gcf(float *gammcf, float a, float x, float *gln);
    void gser(float *gamser, float a, float x, float *gln);
    void nrerror(char error_text[]);
    float gamser,gammcf,gln;

    if (x < 0.0 || a <= 0.0) nrerror("Invalid arguments in routine gammp");
    if (x < (a+1.0)) {              Use the series representation.
        gser(&gamser,a,x,&gln);
        return gamser;
    } else {                        Use the continued fraction representation
        gcf(&gammcf,a,x,&gln);
        return 1.0-gammcf;          and take its complement.
    }
}
```

```
float gammq(float a, float x)
Returns the incomplete gamma function Q(a, x) ≡ 1 − P(a, x).
{
    void gcf(float *gammcf, float a, float x, float *gln);
    void gser(float *gamser, float a, float x, float *gln);
    void nrerror(char error_text[]);
    float gamser,gammcf,gln;

    if (x < 0.0 || a <= 0.0) nrerror("Invalid arguments in routine gammq");
    if (x < (a+1.0)) {              Use the series representation
        gser(&gamser,a,x,&gln);
        return 1.0-gamser;          and take its complement.
    } else {                        Use the continued fraction representation.
        gcf(&gammcf,a,x,&gln);
        return gammcf;
    }
}
```

The argument `gln` is set by both the series and continued fraction procedures to the value $\ln \Gamma(a)$; the reason for this is so that it is available to you if you want to modify the above two procedures to give $\gamma(a, x)$ and $\Gamma(a, x)$, in addition to $P(a, x)$ and $Q(a, x)$ (cf. equations 6.2.1 and 6.2.3).

The functions `gser` and `gcf` which implement (6.2.5) and (6.2.7) are

```
#include <math.h>
#define ITMAX 100
#define EPS 3.0e-7

void gser(float *gamser, float a, float x, float *gln)
Returns the incomplete gamma function P(a, x) evaluated by its series representation as gamser.
Also returns ln Γ(a) as gln.
{
    float gammln(float xx);
```

```
    void nrerror(char error_text[]);
    int n;
    float sum,del,ap;

    *gln=gammln(a);
    if (x <= 0.0) {
        if (x < 0.0) nrerror("x less than 0 in routine gser");
        *gamser=0.0;
        return;
    } else {
        ap=a;
        del=sum=1.0/a;
        for (n=1;n<=ITMAX;n++) {
            ++ap;
            del *= x/ap;
            sum += del;
            if (fabs(del) < fabs(sum)*EPS) {
                *gamser=sum*exp(-x+a*log(x)-(*gln));
                return;
            }
        }
        nrerror("a too large, ITMAX too small in routine gser");
        return;
    }
}
```

```
#include <math.h>
#define ITMAX 100                    Maximum allowed number of iterations.
#define EPS 3.0e-7                   Relative accuracy.
#define FPMIN 1.0e-30                Number near the smallest representable
                                         floating-point number.
void gcf(float *gammcf, float a, float x, float *gln)
```

Returns the incomplete gamma function $Q(a,x)$ evaluated by its continued fraction representation as gammcf. Also returns $\ln\Gamma(a)$ as gln.

```
{
    float gammln(float xx);
    void nrerror(char error_text[]);
    int i;
    float an,b,c,d,del,h;

    *gln=gammln(a);
    b=x+1.0-a;                       Set up for evaluating continued fraction
    c=1.0/FPMIN;                         by modified Lentz's method (§5.2)
    d=1.0/b;                             with b0 = 0.
    h=d;
    for (i=1;i<=ITMAX;i++) {         Iterate to convergence.
        an = -i*(i-a);
        b += 2.0;
        d=an*d+b;
        if (fabs(d) < FPMIN) d=FPMIN;
        c=b+an/c;
        if (fabs(c) < FPMIN) c=FPMIN;
        d=1.0/d;
        del=d*c;
        h *= del;
        if (fabs(del-1.0) < EPS) break;
    }
    if (i > ITMAX) nrerror("a too large, ITMAX too small in gcf");
    *gammcf=exp(-x+a*log(x)-(*gln))*h;    Put factors in front.
}
```

Error Function

The error function and complementary error function are special cases of the incomplete gamma function, and are obtained moderately efficiently by the above procedures. Their definitions are

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{6.2.8}$$

and

$$\mathrm{erfc}(x) \equiv 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{6.2.9}$$

The functions have the following limiting values and symmetries:

$$\mathrm{erf}(0) = 0 \qquad \mathrm{erf}(\infty) = 1 \qquad \mathrm{erf}(-x) = -\mathrm{erf}(x) \tag{6.2.10}$$

$$\mathrm{erfc}(0) = 1 \qquad \mathrm{erfc}(\infty) = 0 \qquad \mathrm{erfc}(-x) = 2 - \mathrm{erfc}(x) \tag{6.2.11}$$

They are related to the incomplete gamma functions by

$$\mathrm{erf}(x) = P\left(\frac{1}{2}, x^2\right) \qquad (x \geq 0) \tag{6.2.12}$$

and

$$\mathrm{erfc}(x) = Q\left(\frac{1}{2}, x^2\right) \qquad (x \geq 0) \tag{6.2.13}$$

We'll put an extra "f" into our routine names to avoid conflicts with names already in some `C` libraries:

```
float erff(float x)
Returns the error function erf(x).
{
    float gammp(float a, float x);

    return x < 0.0 ? -gammp(0.5,x*x) : gammp(0.5,x*x);
}
```

```
float erffc(float x)
Returns the complementary error function erfc(x).
{
    float gammp(float a, float x);
    float gammq(float a, float x);

    return x < 0.0 ? 1.0+gammp(0.5,x*x) : gammq(0.5,x*x);
}
```

If you care to do so, you can easily remedy the minor inefficiency in `erff` and `erffc`, namely that $\Gamma(0.5) = \sqrt{\pi}$ is computed unnecessarily when `gammp` or `gammq` is called. Before you do that, however, you might wish to consider the following routine, based on Chebyshev fitting to an inspired guess as to the functional form:

```
#include <math.h>

float erfcc(float x)
Returns the complementary error function erfc(x) with fractional error everywhere less than
1.2 × 10^-7.
{
    float t,z,ans;

    z=fabs(x);
    t=1.0/(1.0+0.5*z);
    ans=t*exp(-z*z-1.26551223+t*(1.00002368+t*(0.37409196+t*(0.09678418+
        t*(-0.18628806+t*(0.27886807+t*(-1.13520398+t*(1.48851587+
        t*(-0.82215223+t*0.17087277)))))))));
    return x >= 0.0 ? ans : 2.0-ans;
}
```

There are also some functions of *two* variables that are special cases of the incomplete gamma function:

Cumulative Poisson Probability Function

$P_x(< k)$, for positive x and integer $k \geq 1$, denotes the *cumulative Poisson probability* function. It is defined as the probability that the number of Poisson random events occurring will be between 0 and $k-1$ *inclusive*, if the expected mean number is x. It has the limiting values

$$P_x(< 1) = e^{-x} \qquad P_x(< \infty) = 1 \tag{6.2.14}$$

Its relation to the incomplete gamma function is simply

$$P_x(< k) = Q(k, x) = \texttt{gammq}\,(k, x) \tag{6.2.15}$$

Chi-Square Probability Function

$P(\chi^2|\nu)$ is defined as the probability that the observed chi-square for a correct model should be less than a value χ^2. (We will discuss the use of this function in Chapter 15.) Its complement $Q(\chi^2|\nu)$ is the probability that the observed chi-square will exceed the value χ^2 by chance *even* for a correct model. In both cases ν is an integer, the number of degrees of freedom. The functions have the limiting values

$$P(0|\nu) = 0 \qquad P(\infty|\nu) = 1 \tag{6.2.16}$$

$$Q(0|\nu) = 1 \qquad Q(\infty|\nu) = 0 \tag{6.2.17}$$

and the following relation to the incomplete gamma functions,

$$P(\chi^2|\nu) = P\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \texttt{gammp}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \tag{6.2.18}$$

$$Q(\chi^2|\nu) = Q\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \texttt{gammq}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \tag{6.2.19}$$

CITED REFERENCES AND FURTHER READING:

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, vol. 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapters 6, 7, and 26.

Pearson, K. (ed.) 1951, *Tables of the Incomplete Gamma Function* (Cambridge: Cambridge University Press).

6.3 Exponential Integrals

The standard definition of the exponential integral is

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt, \qquad x > 0, \quad n = 0, 1, \dots \tag{6.3.1}$$

The function defined by the principal value of the integral

$$\mathrm{Ei}(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}\, dt = \int_{-\infty}^x \frac{e^t}{t}\, dt, \qquad x > 0 \tag{6.3.2}$$

is also called an exponential integral. Note that $\mathrm{Ei}(-x)$ is related to $-E_1(x)$ by analytic continuation.

The function $E_n(x)$ is a special case of the incomplete gamma function

$$E_n(x) = x^{n-1}\Gamma(1-n, x) \tag{6.3.3}$$

We can therefore use a similar strategy for evaluating it. The continued fraction — just equation (6.2.6) rewritten — converges for all $x > 0$:

$$E_n(x) = e^{-x}\left(\frac{1}{x+}\,\frac{n}{1+}\,\frac{1}{x+}\,\frac{n+1}{1+}\,\frac{2}{x+}\cdots\right) \tag{6.3.4}$$

We use it in its more rapidly converging even form,

$$E_n(x) = e^{-x}\left(\frac{1}{x+n-}\,\frac{1\cdot n}{x+n+2-}\,\frac{2(n+1)}{x+n+4-}\,\cdots\right) \tag{6.3.5}$$

The continued fraction only really converges fast enough to be useful for $x \gtrsim 1$. For $0 < x \lesssim 1$, we can use the series representation

$$E_n(x) = \frac{(-x)^{n-1}}{(n-1)!}[-\ln x + \psi(n)] - \sum_{\substack{m=0 \\ m\neq n-1}}^{\infty} \frac{(-x)^m}{(m-n+1)m!} \tag{6.3.6}$$

The quantity $\psi(n)$ here is the digamma function, given for integer arguments by

$$\psi(1) = -\gamma, \qquad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \tag{6.3.7}$$